

Compressed Sensing MRI

M229 Advanced Topics in MRI

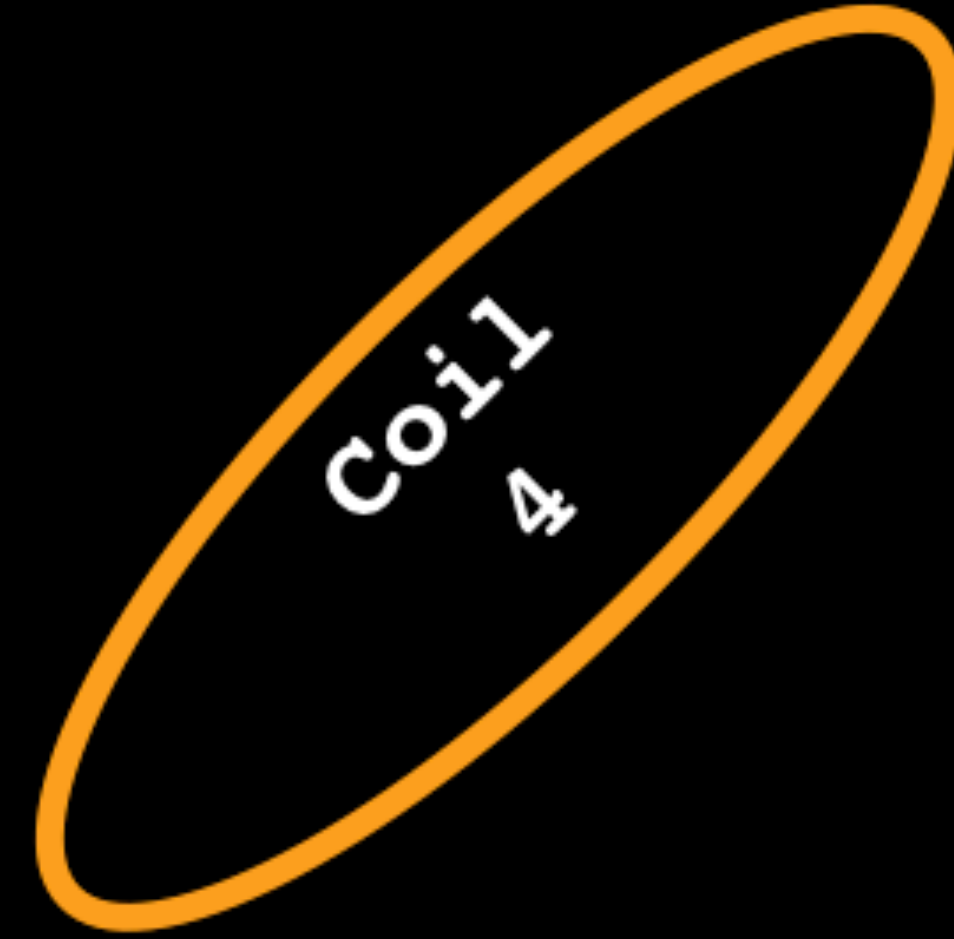
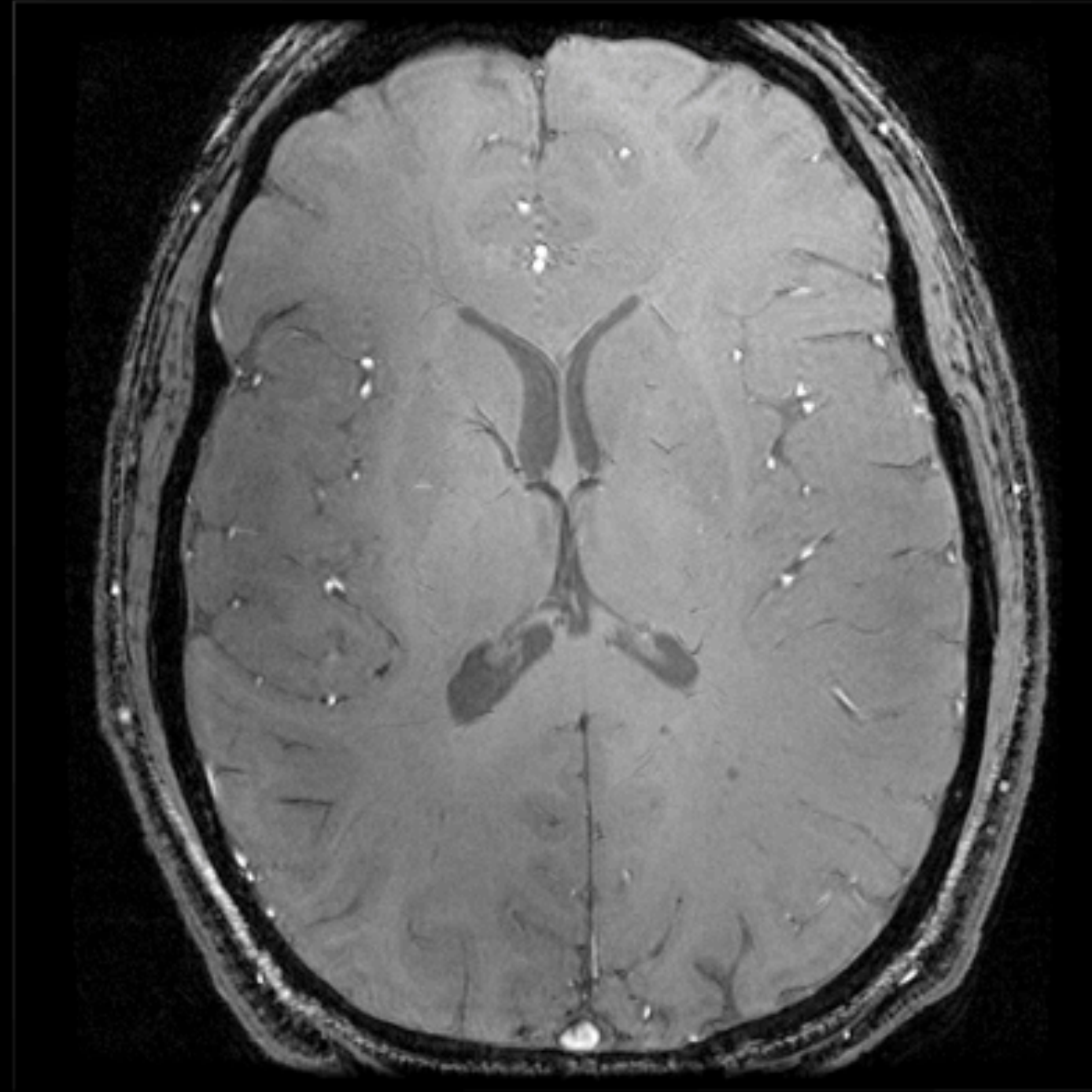
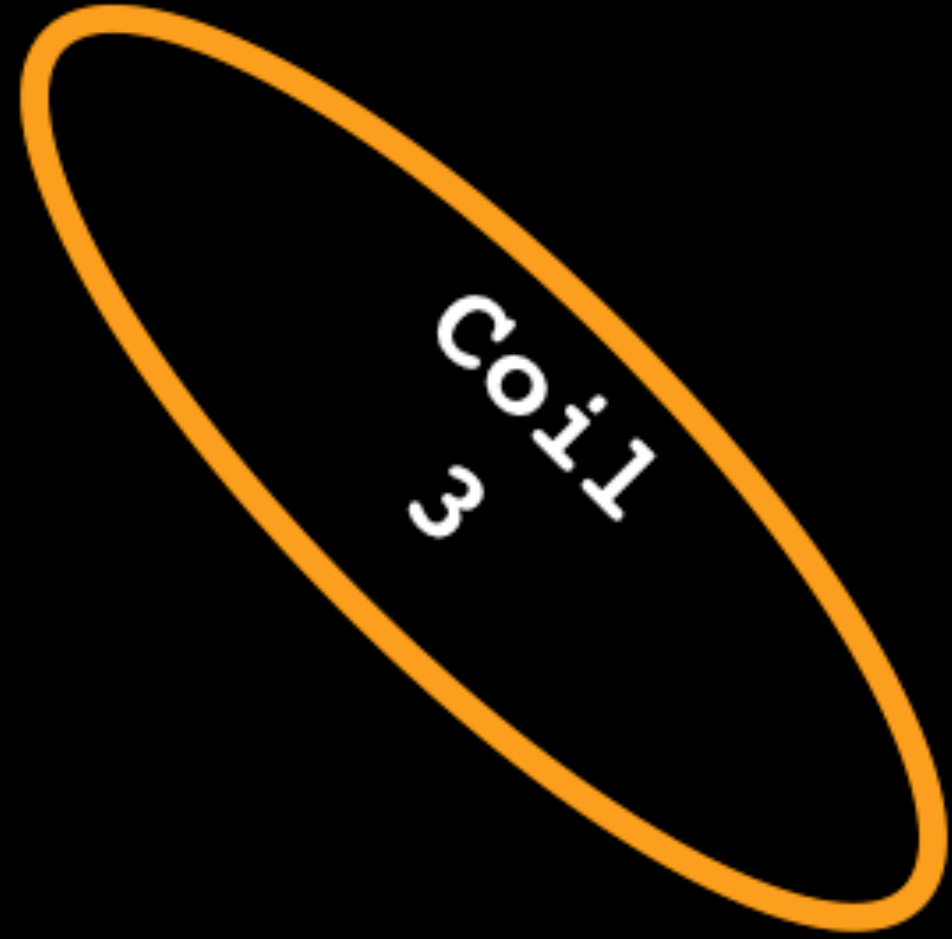
Shu-Fu Shih

5/16/2023

Today's topics

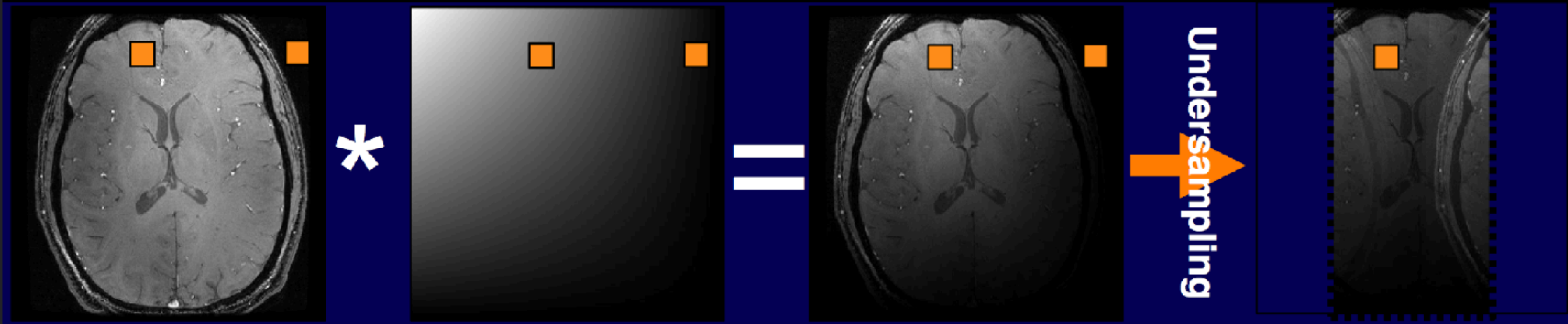
- k-Space properties review
- Compressed sensing MRI (with code examples)
 - Sparse representation
 - Incoherent artifacts
 - Nonlinear reconstruction
- Compressed sensing MRI applications

Multi-coil Arrays

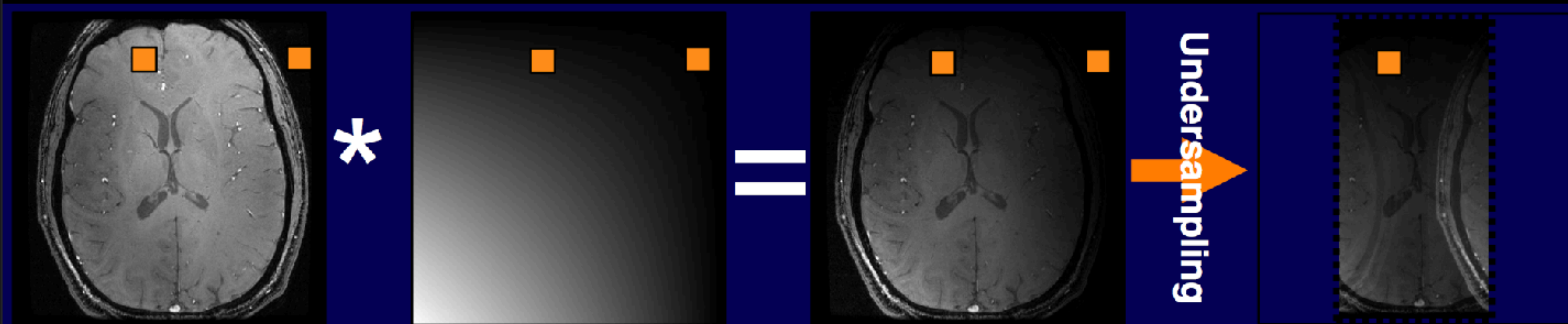


Cartesian SENSE

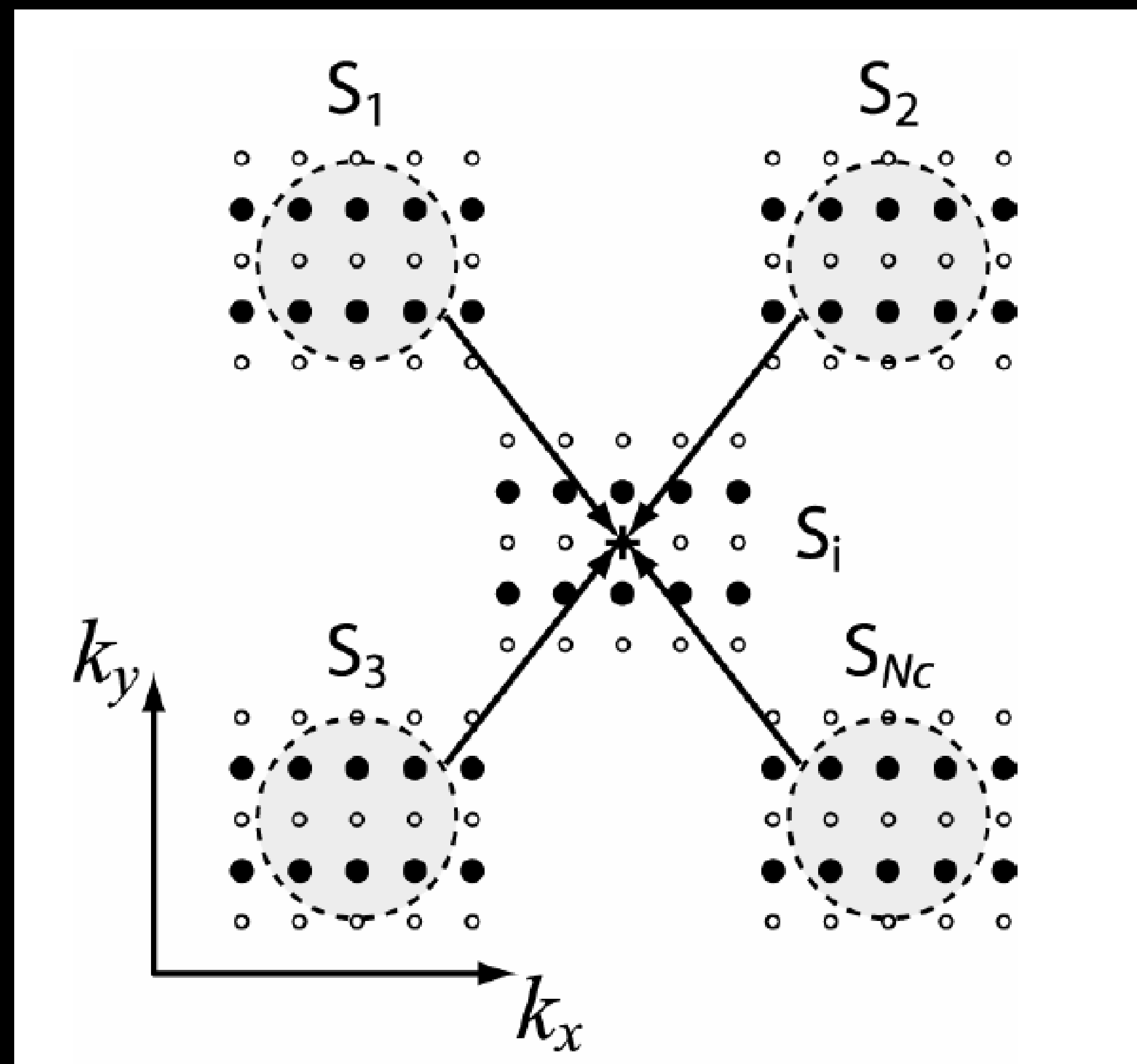
$$m_1(\vec{x}_1) = C_1(\vec{x}_1)m(\vec{x}_1) + C_1(\vec{x}_2)m(\vec{x}_2)$$



$$m_2(\vec{x}_1) = C_2(\vec{x}_1)m(\vec{x}_1) + C_2(\vec{x}_2)m(\vec{x}_2)$$



GRAPPA - Synthesis



Review: Parallel imaging reconstruction

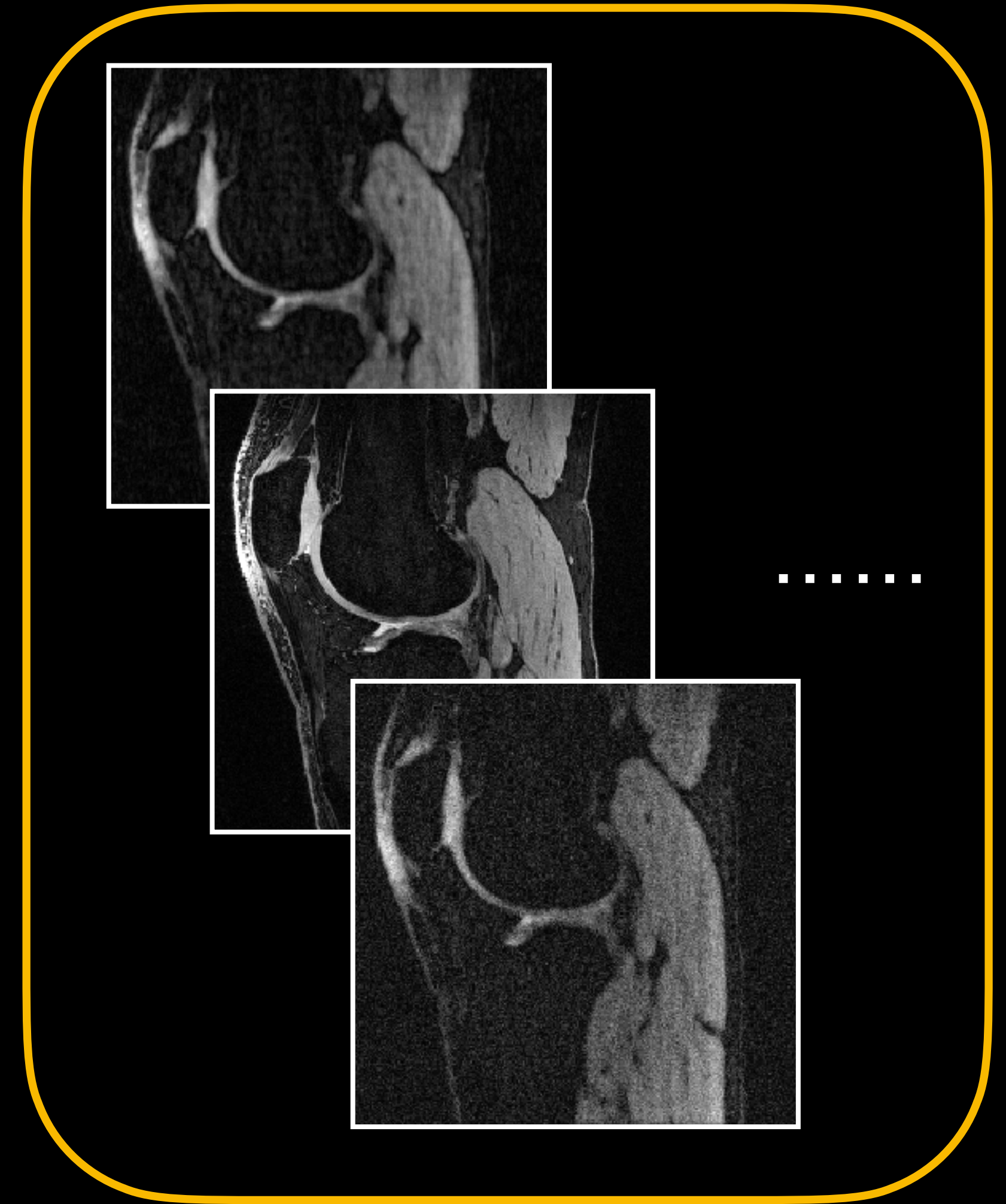
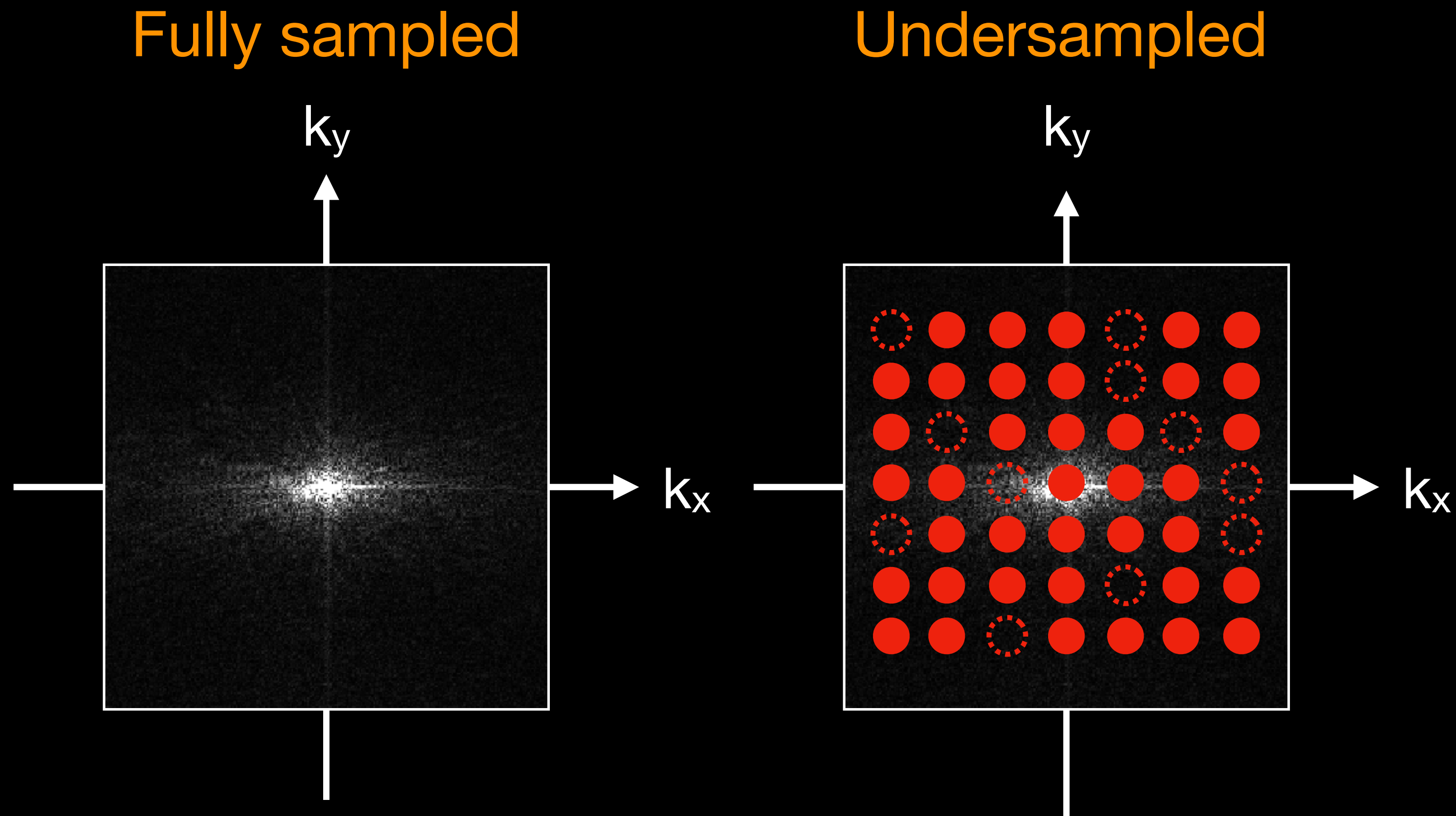
- Parallel imaging utilizes information from multiple coils to accelerate MRI
- Different parallel imaging approaches:
 - SENSE (image-based)
 - GRAPPA (k-space-based)
 - ...

Accelerated MRI

- MRI acquisition time is limited by physics and hardware constraints
- MRI scans can be accelerated by acquiring undersampled k-space data followed by advanced reconstruction
- Accelerated MRI approaches
 - (1) Parallel imaging
 - Use information from multiple coils
 - (2) Compressed sensing
 - Use sparsity constraints as prior information
 - (3) Deep learning
 - Use a nonlinear neural network trained with a large dataset
 - ... and more

Underdetermined system

Images with the same undersampled k-space data



Use prior information about the images to help us solve the underdetermined problem

Compressed sensing MRI

- Compressed sensing MRI can reconstruct an image with high fidelity from undersampled k-space data given
 - (1) the image has transform sparsity (or a **sparse representation** in some transform domain)
 - (2) the k-space sampling pattern generates **incoherent artifacts** in the sparse transform domain
- Compressed sensing MRI usually involves a **nonlinear reconstruction** method to recover the image

Sparse representation

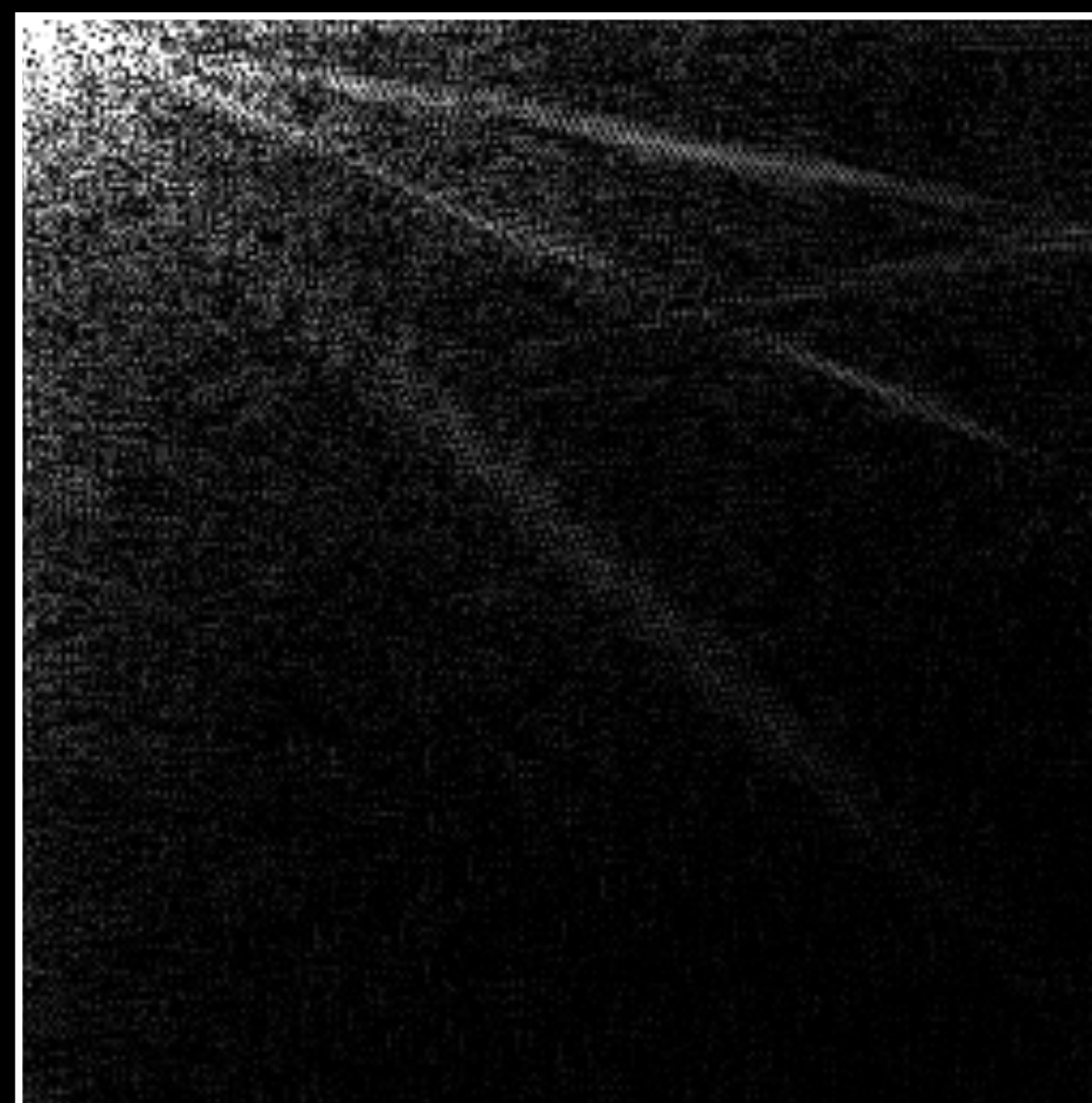
- Many images have a sparse representation in some transform domain
- Example 1: Discrete cosine transform (DCT)
 - JPEG uses DCT for image compression

$$X_k = \sum_{n=0}^{N-1} x_n \cos \left[\frac{\pi}{N} \left(n + \frac{1}{2} \right) k \right] \quad \text{for } k = 0, \dots, N-1$$

Original image



2D DCT coefficients



Compressed image (3.7-fold)
by preserving large DCT coefficients



See code example 01

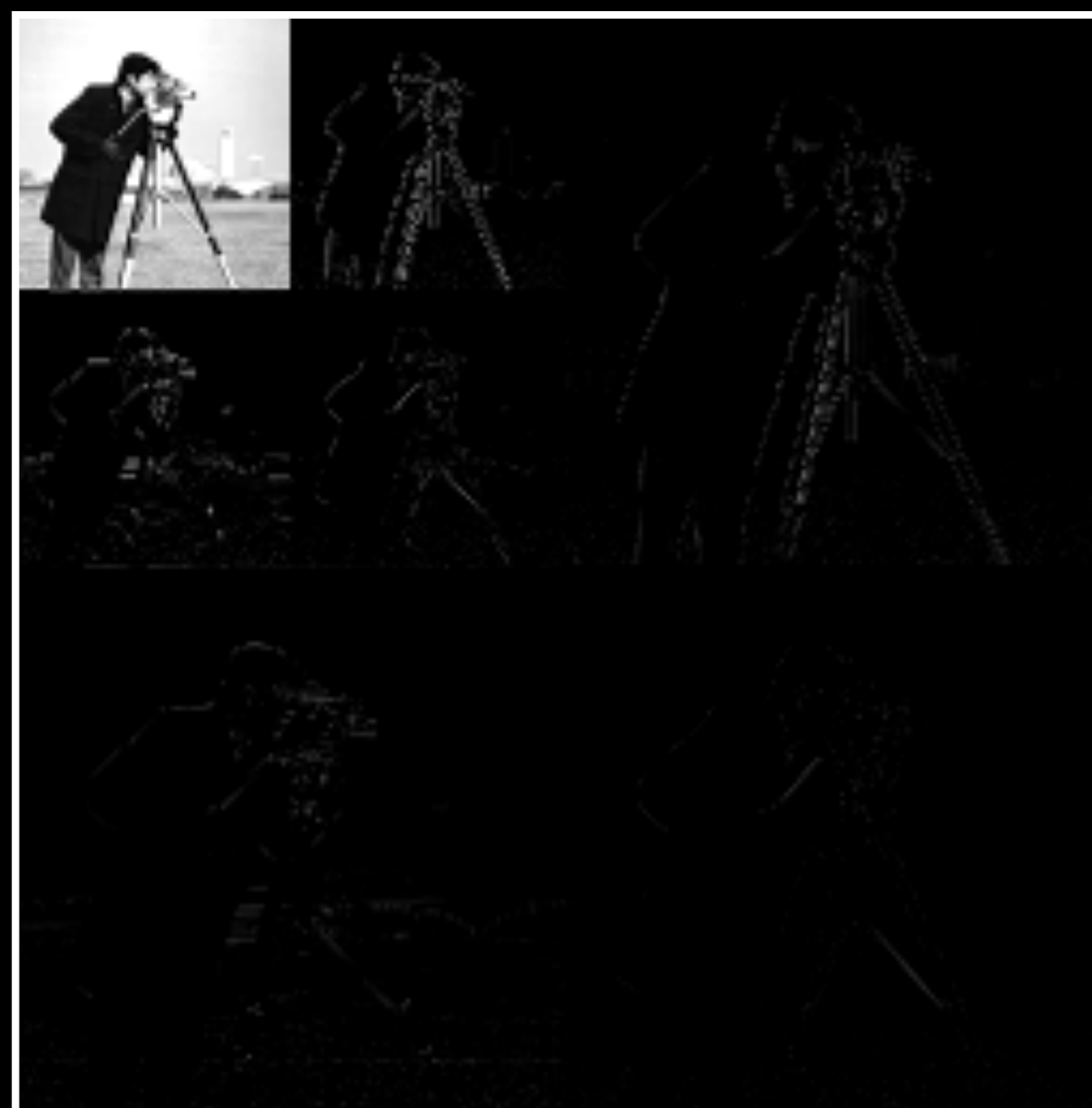
Sparse representation

- Example 2: Wavelet transform
 - JPEG 2000 uses Wavelet transform for image compression

Original image



2D Wavelet coefficients



Compressed image (5.3-fold)
by preserving large Wavelet coefficients

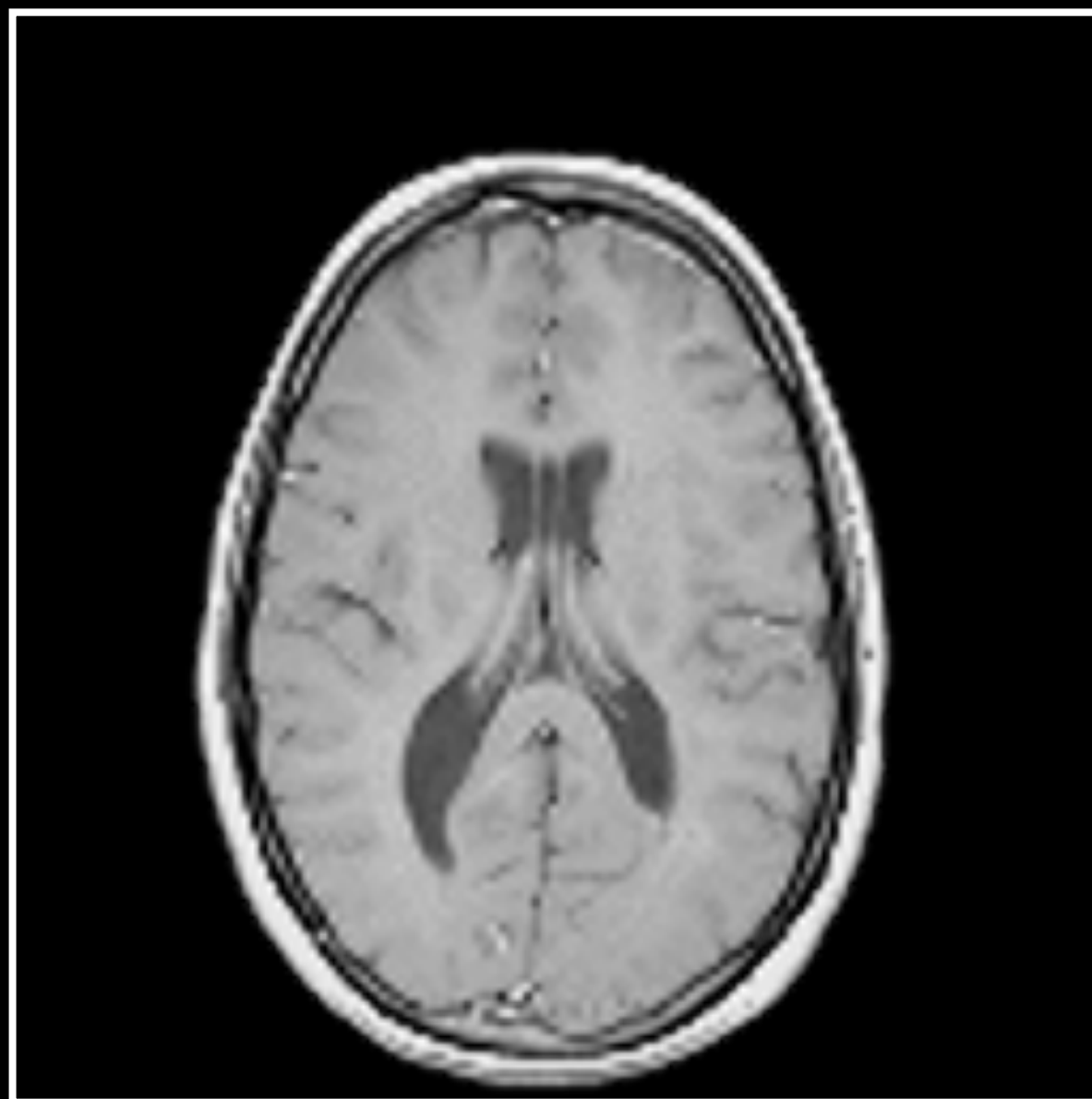


See code example 02

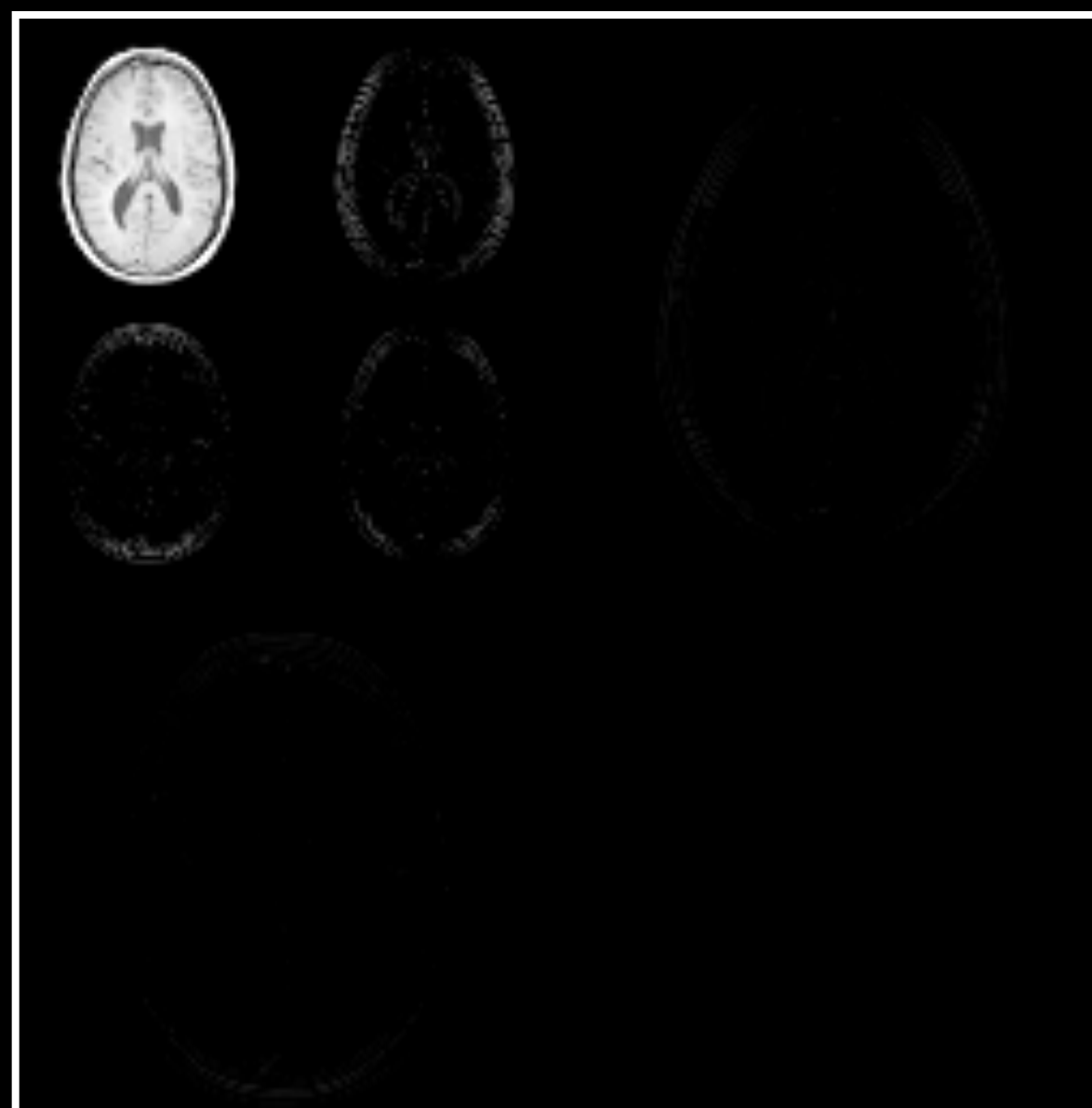
Sparse representation

- Example 3: Wavelet transform for a brain image

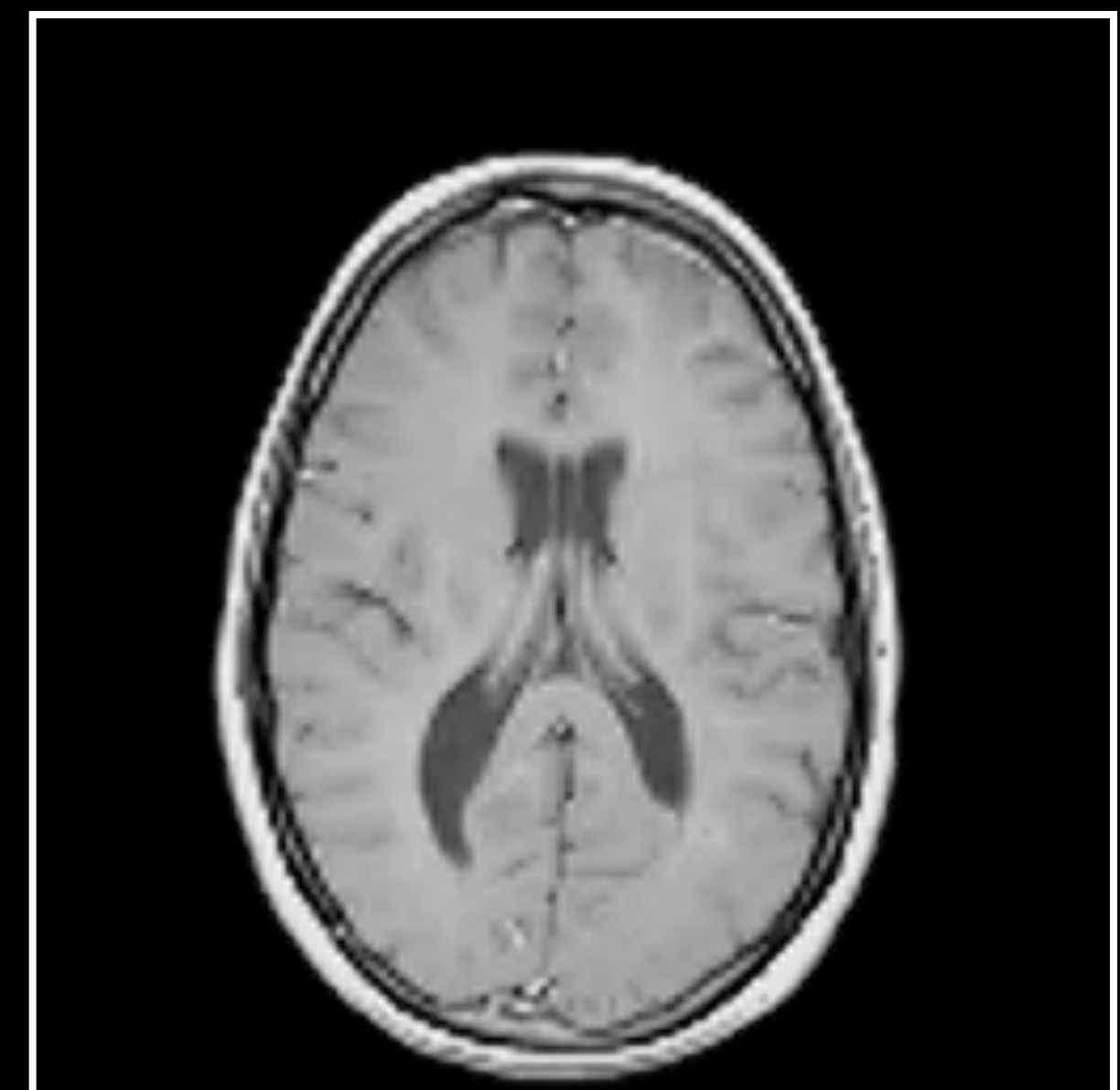
Original image



2D Wavelet coefficients



Compressed image (4.8-fold)
by preserving large Wavelet coefficients

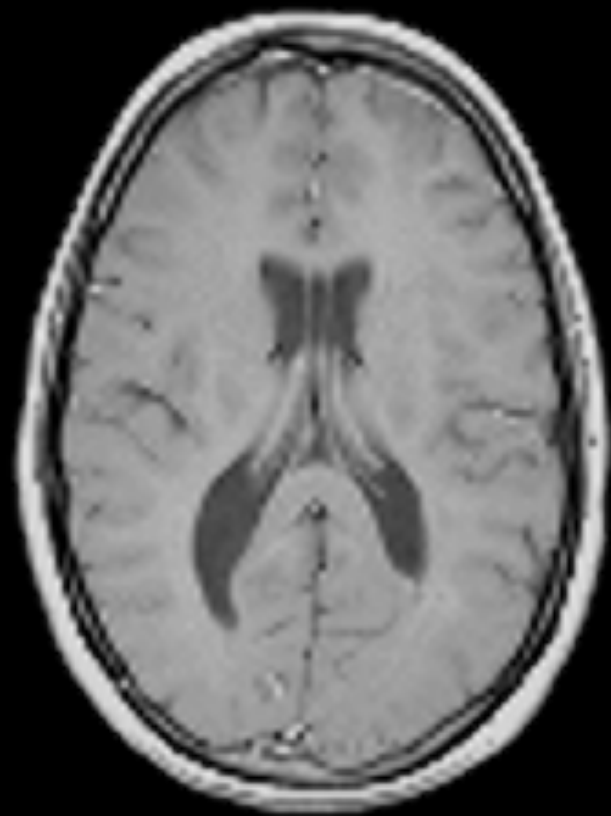


[See code example 03](#)

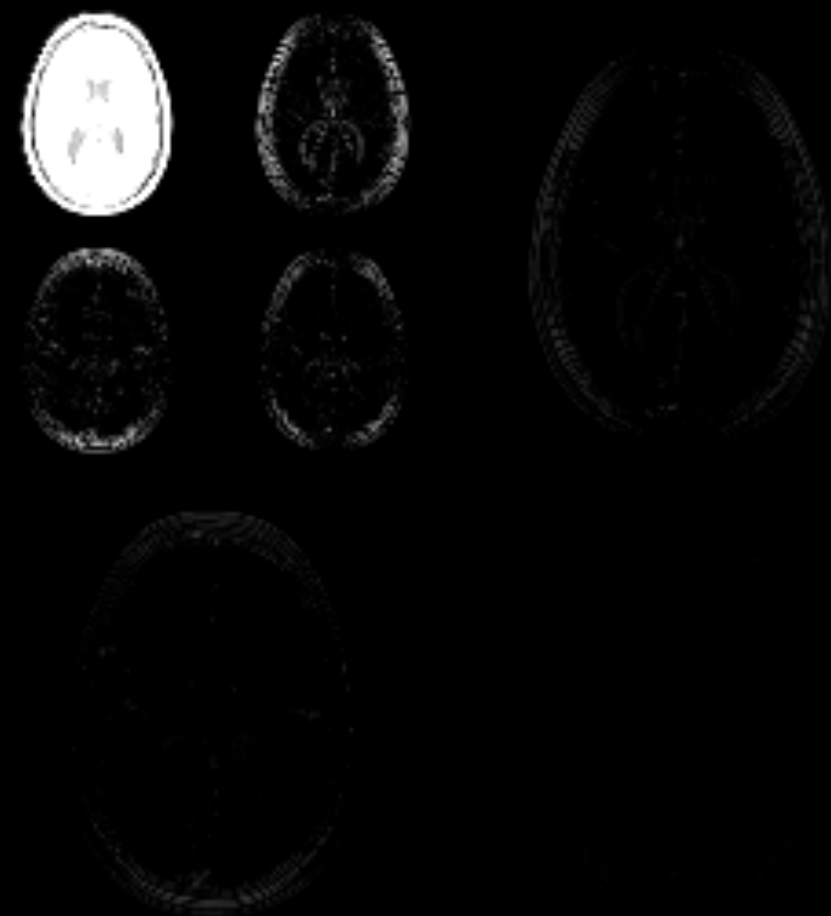
Sparse representation

- Many images have a sparse representation in some transform domain

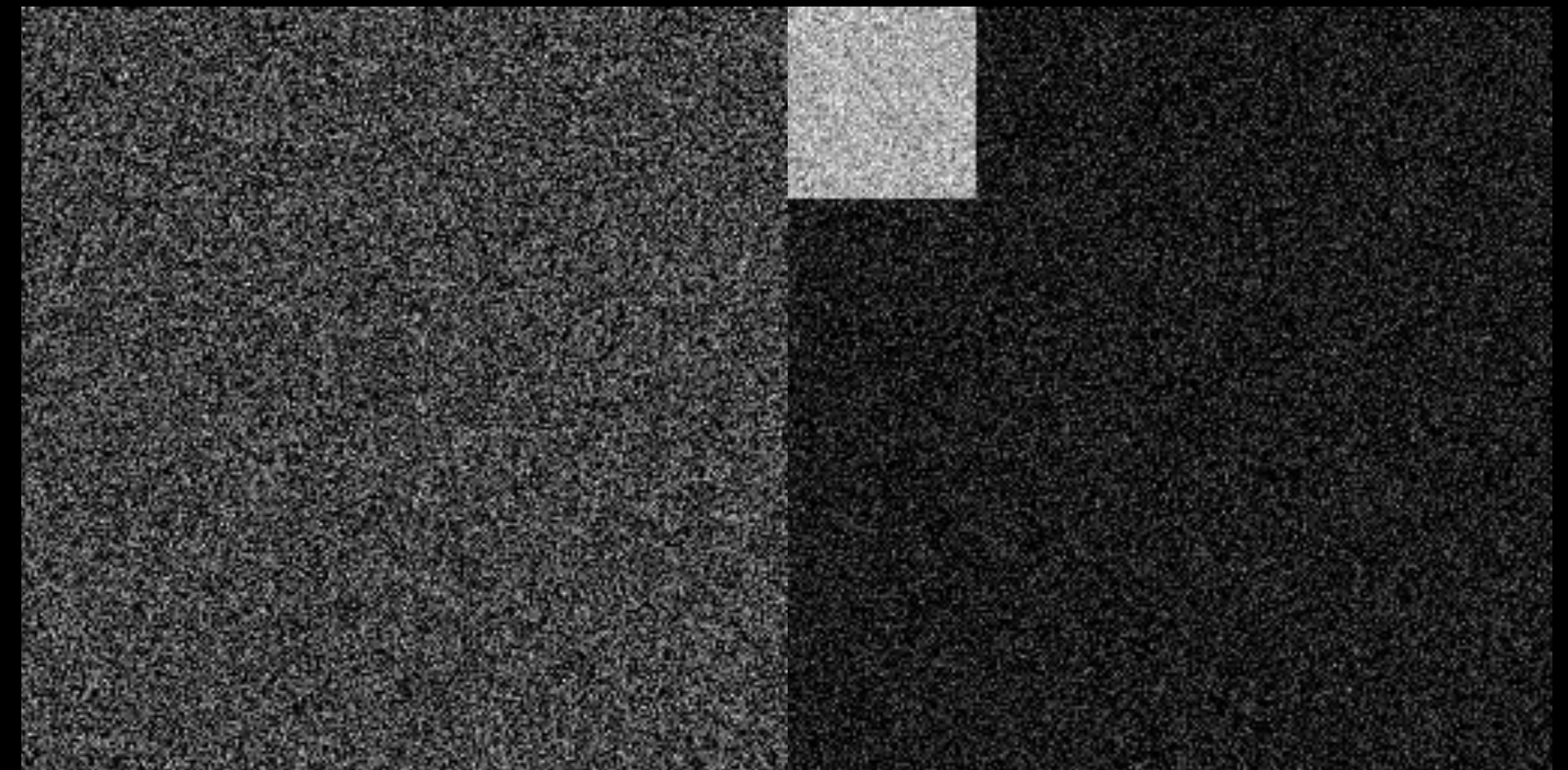
Brain image



2D Wavelet coefficients
of a brain image



Noisy image

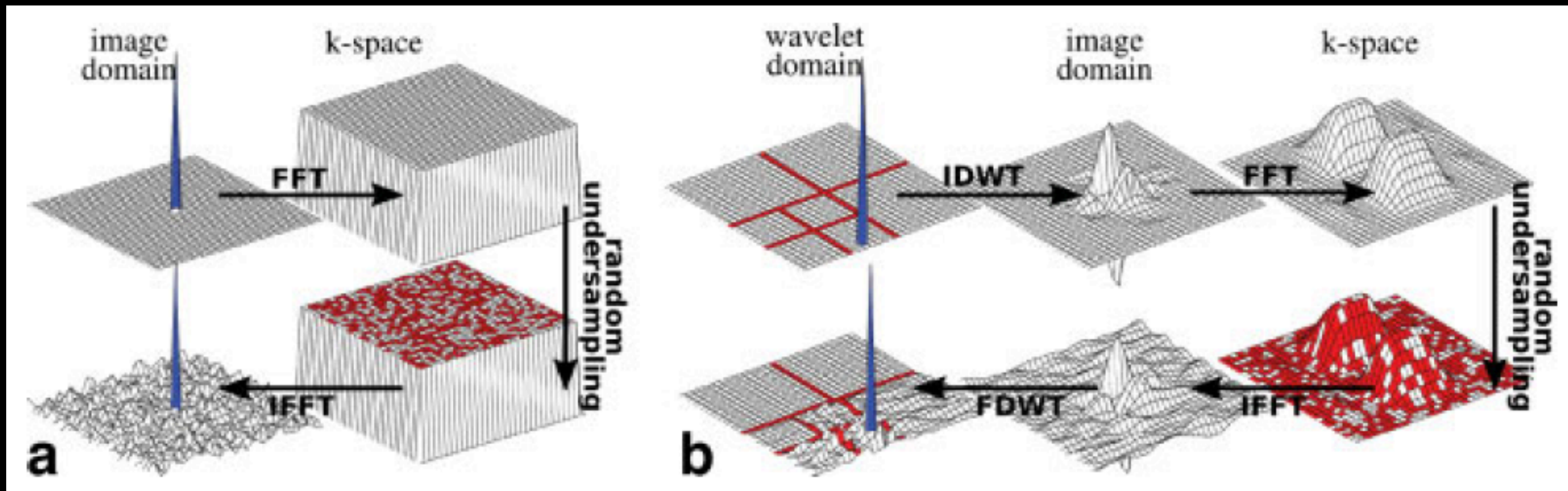


2D Wavelet coefficients
of a noisy image

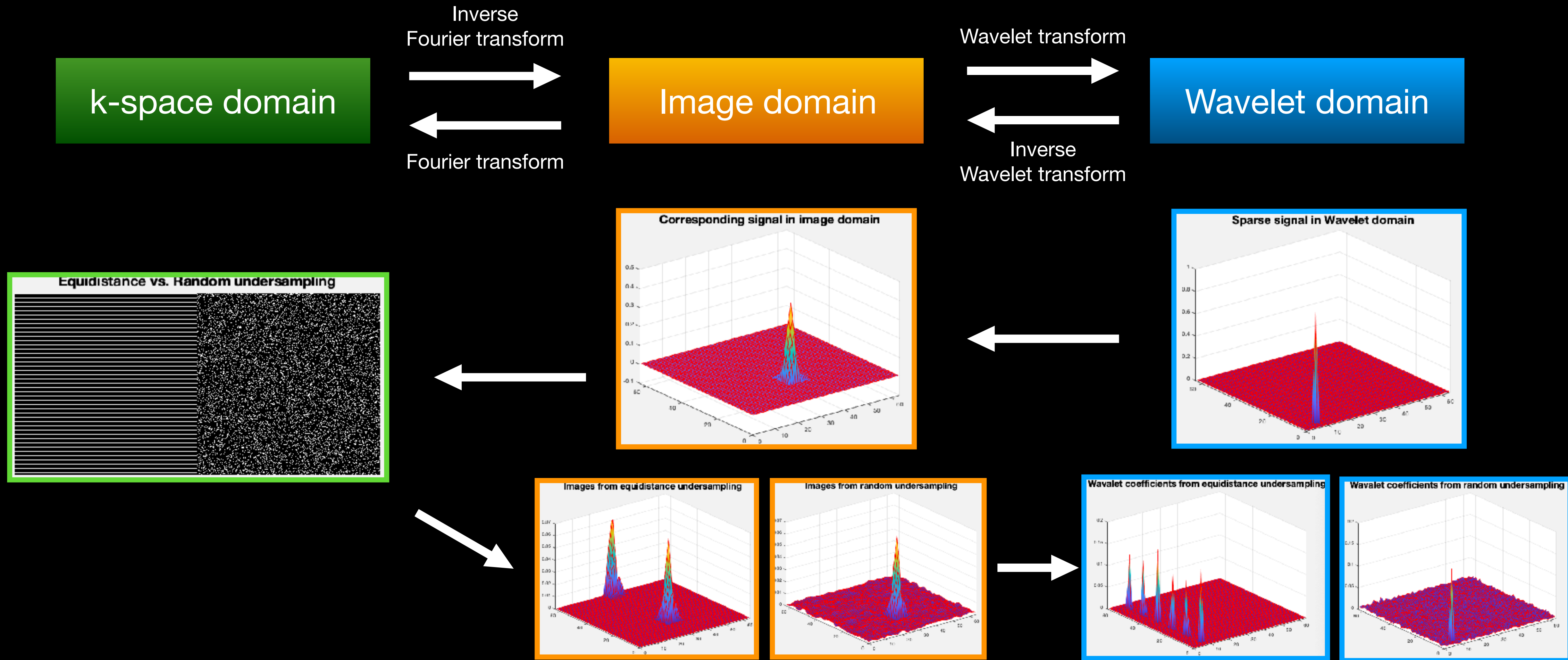
[See code example 04](#)

Incoherent artifacts

- The second requirement for CS MRI is that the undersampling pattern should generate incoherent artifacts in the sparse transform domain



Incoherent artifacts



See code example 05

L0, L1 and L2 norm

- Vector norm: a method to measure the length of a vector
- L_0 norm ($\|x\|_0$): number of non-zero entries
- L_1 norm ($\|x\|_1$): sum of absolute values of the entries
$$\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$$
- L_2 norm ($\|x\|_2$): square root of sum of squared values of the entries

$$\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_n|^2}$$

L0, L1 and L2 norm

$$v_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 2 \\ -3 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 2 \\ -4 \\ 2 \end{bmatrix}$$

- $\|v_1\|_2 = \sqrt{38}$
- $\|v_1\|_1 = 10$
- $\|v_1\|_0 = 3$

- $\|v_2\|_2 = \sqrt{38}$
- $\|v_2\|_1 = 14$
- $\|v_2\|_0 = 6$

- Two vectors with similar energy (L₂ norm) can have different levels of sparsity (L₁ norm)

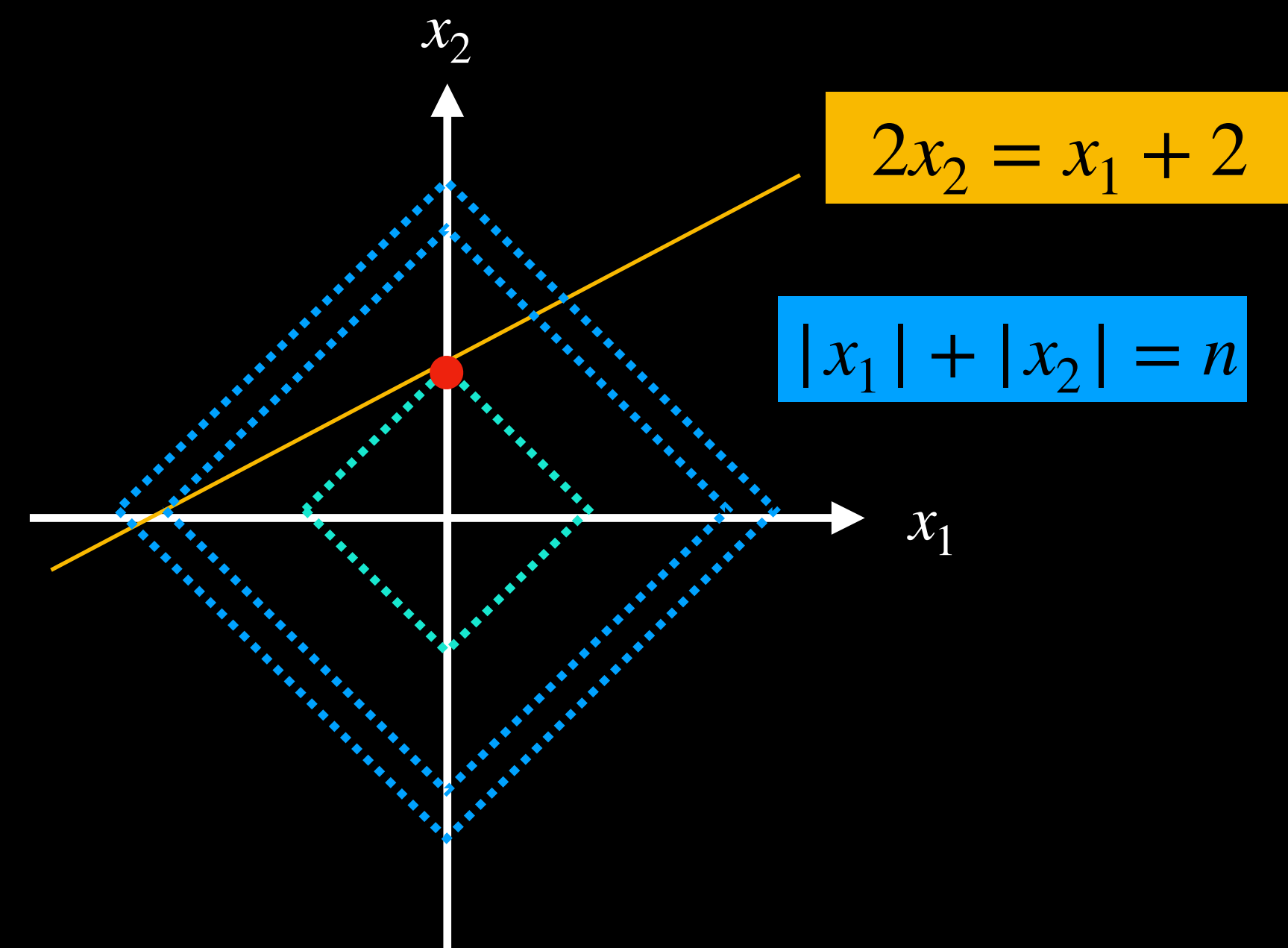
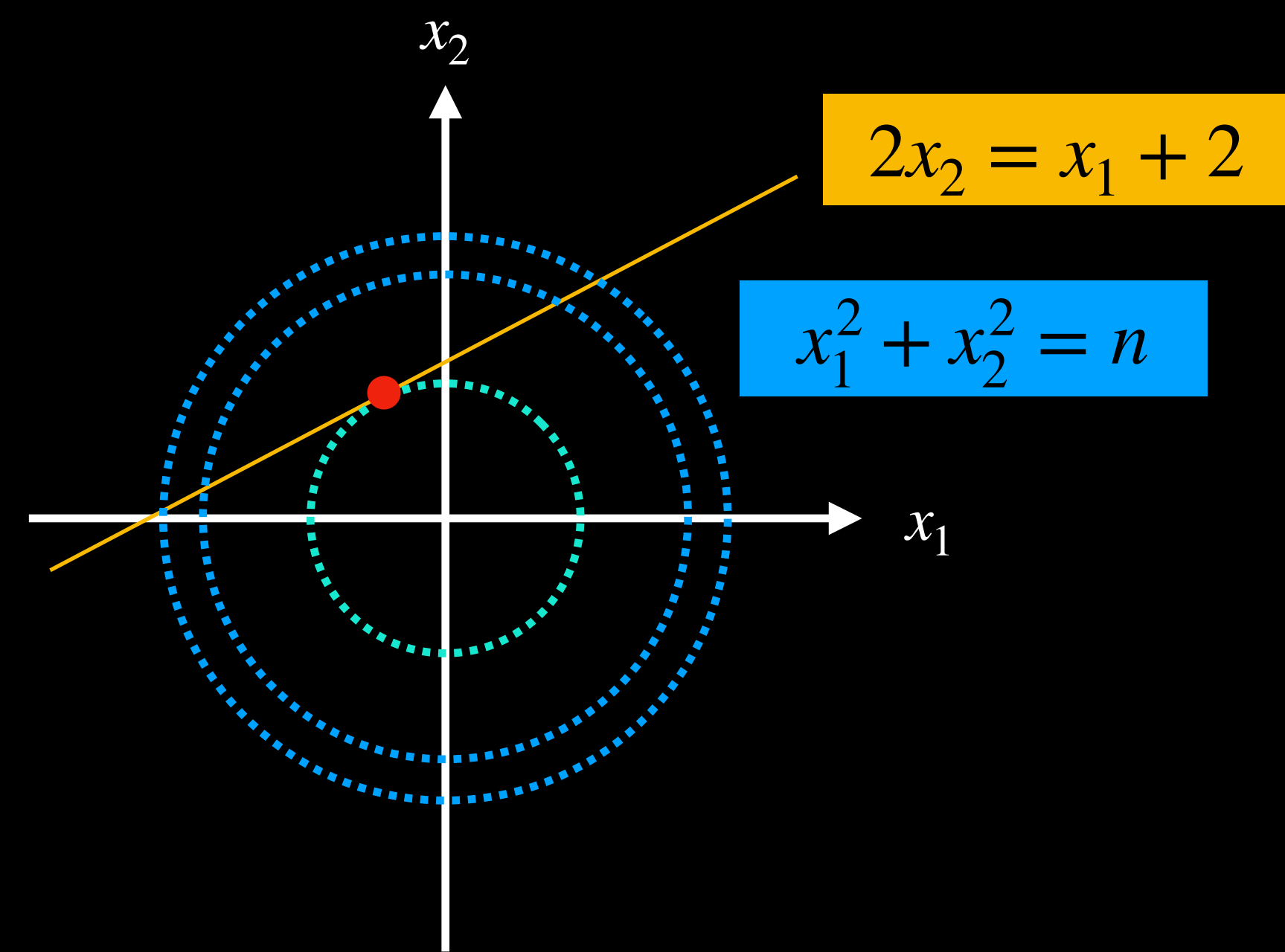
Exercises

- Suppose we have a 2D vector $x = [x_1, x_2]$

- Exercise 1:
$$\begin{aligned} \operatorname{argmin}_x \quad & \|x\|_2 \\ \text{s.t.} \quad & 2x_2 = x_1 + 2 \end{aligned}$$

- Exercise 2:
$$\begin{aligned} \operatorname{argmin}_x \quad & \|x\|_1 \\ \text{s.t.} \quad & 2x_2 = x_1 + 2 \end{aligned}$$

- Example 3:
$$\begin{aligned} \operatorname{argmin}_x \quad & \|x\|_0 \\ \text{s.t.} \quad & 2x_2 = x_1 + 2 \end{aligned}$$



Exercises

- L_2 norm minimization: Find a solution with smallest energy
- L_1 and L_0 norm minimization: Find a sparse solution

Mathematical formulation

Our goal: Find an image that has the sparsest coefficients in the Wavelet domain and the image is consistent with the undersampled k-space data

Turn into an optimization problem



$$\begin{aligned} & \mathit{argmin}_x \quad \| Wx \|_0 \\ & \text{subject to} \quad \| Fx - y \|_2 < \epsilon \end{aligned}$$

Convex relaxation using L1 norm



$$\begin{aligned} & \mathit{argmin}_x \quad \| Wx \|_1 \\ & \text{subject to} \quad \| Fx - y \|_2 < \epsilon \end{aligned}$$

W: Wavelet transform operator
x: reconstructed image
F: Fourier transform operator
y: acquired undersampled k-space data

Mathematical formulation

W: Wavelet transform operator
x: reconstructed image
F: Fourier transform operator
y: acquired undersampled k-space data
 λ : regularization parameter
U: k-space sampling pattern

$$\begin{aligned} & \operatorname{argmin}_x \quad \| Wx \|_1 \\ & \text{subject to} \quad \| Fx - y \|_2 < \epsilon \end{aligned}$$

Use Lagrangian form



$$\operatorname{argmin}_x \quad \| Fx - y \|_2^2 + \lambda \| Wx \|_1$$

Explicitly include an
sampling operator



$$\operatorname{argmin}_x \quad \| UFx - y \|_2^2 + \lambda \| Wx \|_1$$

Mathematical formulation

W: Wavelet transform operator
x: reconstructed image
F: Fourier transform operator
y: acquired undersampled k-space data
 λ : regularization parameter
U: k-space sampling pattern

$$\begin{aligned} & \underset{x}{\operatorname{argmin}} \quad \| Wx \|_1 \\ & \text{subject to} \quad \| Fx - y \|_2 < \epsilon \end{aligned}$$

Use Lagrangian form



$$\underset{x}{\operatorname{argmin}} \quad \| Fx - y \|_2^2 + \lambda \| Wx \|_1$$

Explicitly include an
sampling operator



$$\underset{x}{\operatorname{argmin}} \quad \| UFx - y \|_2^2 + \lambda \| Wx \|_1$$

Cost function

Optimization algorithm

- Solving $\min \|UFx - y\|_2^2 + \lambda \|Wx\|_1$ is non-trivial since the cost function is not smoothed at $Wx=0$
- Different approaches have been used to solve $\min \|UFx - y\|_2^2 + \lambda \|Wx\|_1$
 - Conjugate gradient descent¹
 - ADMM^{2,3}
 - Primal-dual algorithm⁴
 - ...

[1] Lustig et al., *Magn Reson Med.* 2007;58(6):1182-95

[2] Wang et al., *SIAM J Imag Sci.* 2008;1(3):248-72

[3] Ramani et al., *IEEE Trans Med Imaging.* 2011;30(3):694-706

[4] Chambolle et al., *J Math Imaging Vision.* 2011;40(1):120-45

Optimization algorithm

- Conjugate gradient descent

$$\operatorname{argmin}_m f(m) = \left\| U F m - y \right\|_2^2 + \lambda \left\| W x \right\|_1$$

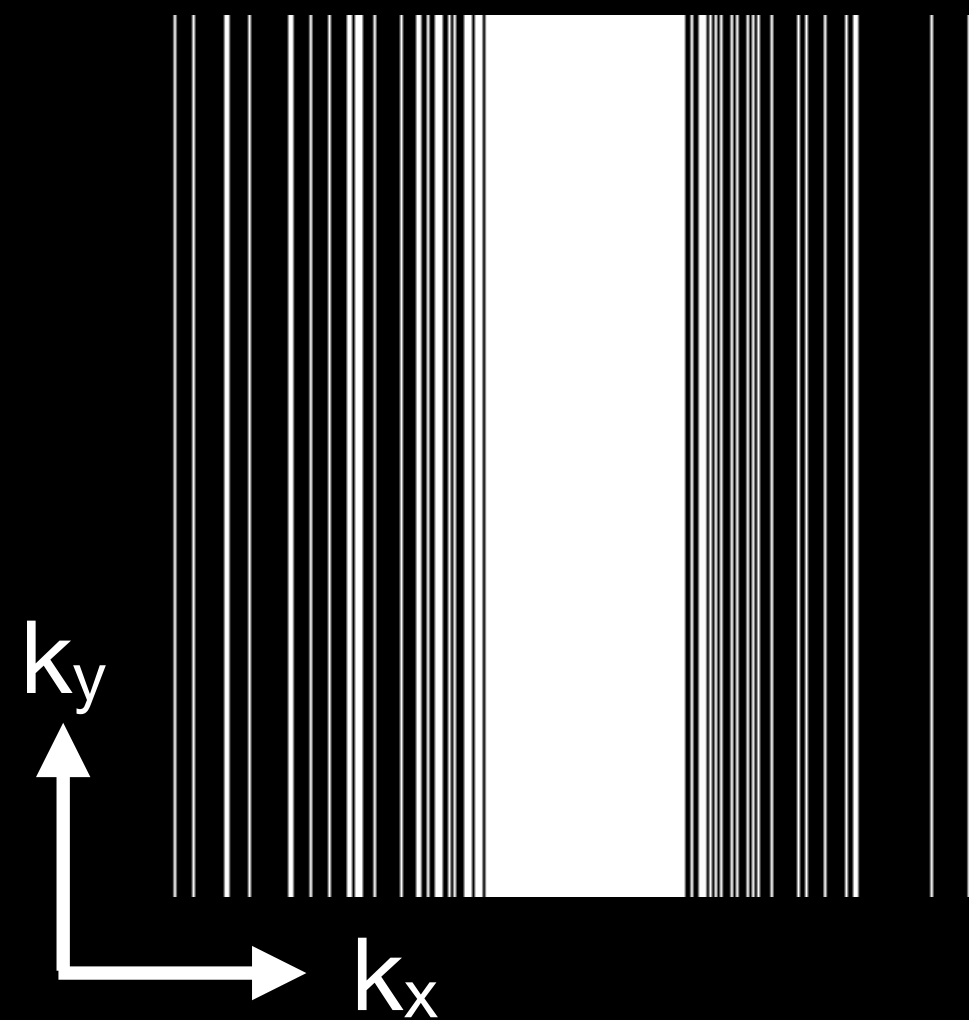
```
% Initialization
k = 0; m = 0; g0 = ∇f(m0); Δm0 = -g0
% Iterations
while (||gk||2 < TolGrad and k > maxIter) {
    % Backtracking line-search
    t = 1; while (f(mk+tΔmk) > f(mk)+αt·Real(gk*Δmk))
        {t = βt}
    mk+1 = mk + tΔmk
    gk+1 = ∇f(mk+1)
    γ = ||gk+1||2 / ||gk||2
    Δmk+1 = -gk+1 + γ Δmk
    k = k + 1 }
```

g_k : gradient at k^{th} iteration
 m_k : updated image result at k^{th} iteration
TolGrad: stopping criteria
MaxIter: stopping criteria on iterations
 α, β : line search parameters

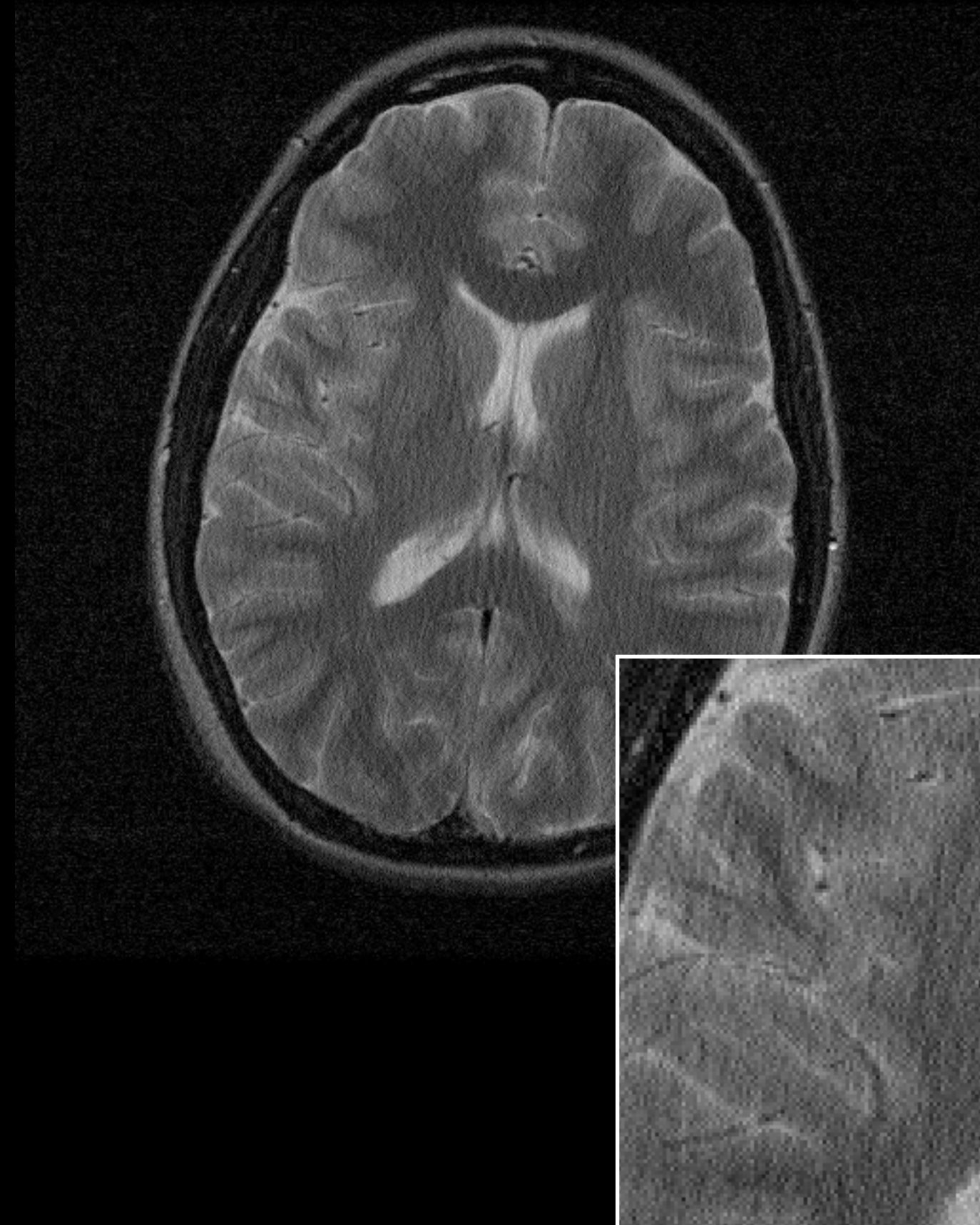
Compressed sensing MRI

- Let's run codes to reconstruct images using compressed sensing MRI...
(see [*code example 06*](#))

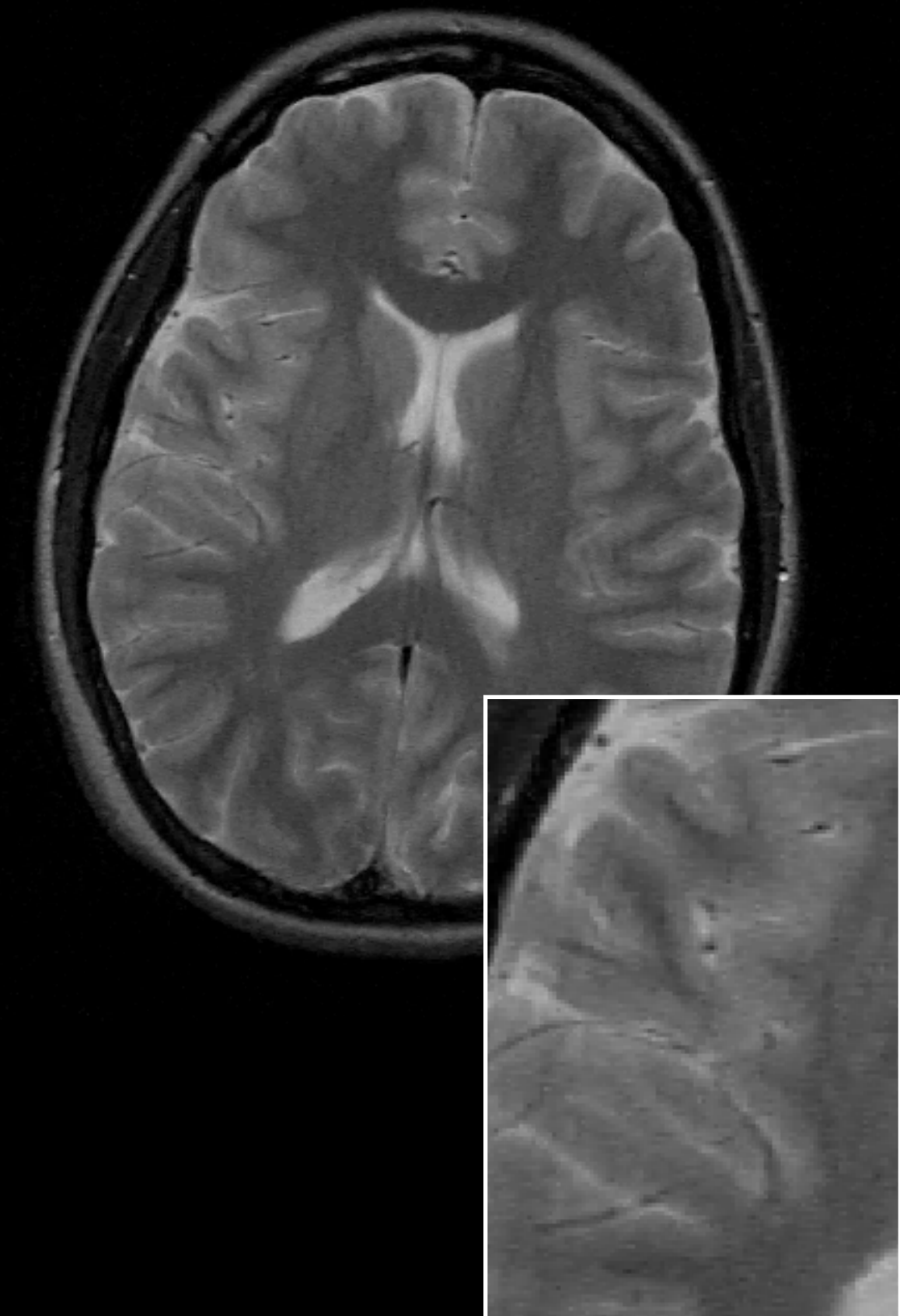
Undersampling mask



Zero-filled



Compressed sensing reconstruction



Compressed sensing MRI

- Compressed sensing MRI can reconstruct an image with high fidelity from undersampled k-space data given
 - (1) the image has transform sparsity (or a **sparse representation** in some transform domain)
 - (2) the k-space sampling pattern generates **incoherent artifacts** in the sparse transform domain
- Compressed sensing MRI usually involves a **nonlinear reconstruction** method to recover the image

Choice of regularization parameters

$$\operatorname{argmin}_x \left\| UFx - y \right\|_2^2 + \lambda \left\| Wx \right\|_1$$

- Many compressed sensing methods require manually tuning of regularization parameters.
- Larger weights on the sparsity term (larger λ):
 - Better suppression on noise or artifacts / Improved perceived SNR
 - Features more likely to be over-smoothed / Resulting in images with artificial appearance
- The regularization parameter is dataset-dependent
- Methods for automatic regularization parameters selection have been investigated

Compressed sensing + Parallel imaging

- Parallel imaging: Use information from multiple coils (e.g., coil sensitivity in SENSE reconstruction)
- Compressed sensing: Use sparsity constraints
- Combination of these two techniques:

$$\mathit{argmin}_x \left\| UFSx - y \right\|_2^2 + \lambda \left\| Wx \right\|_1$$

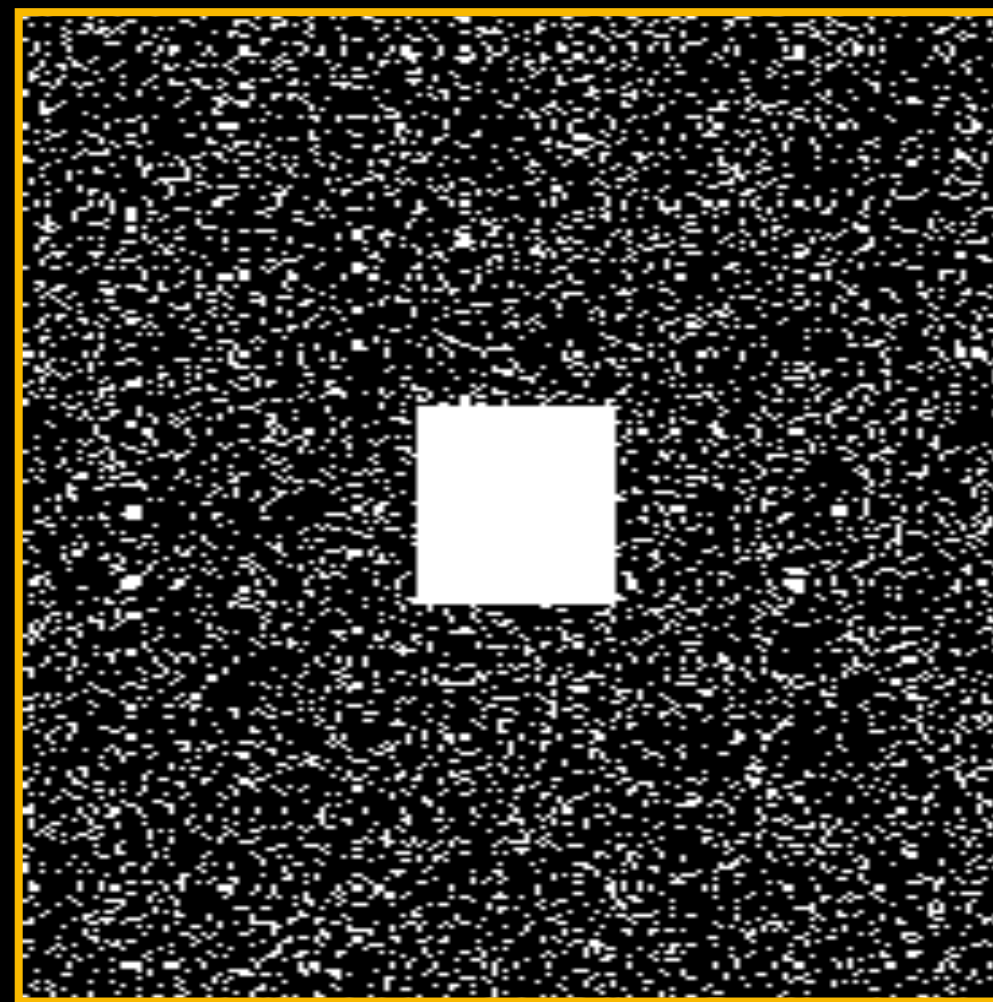
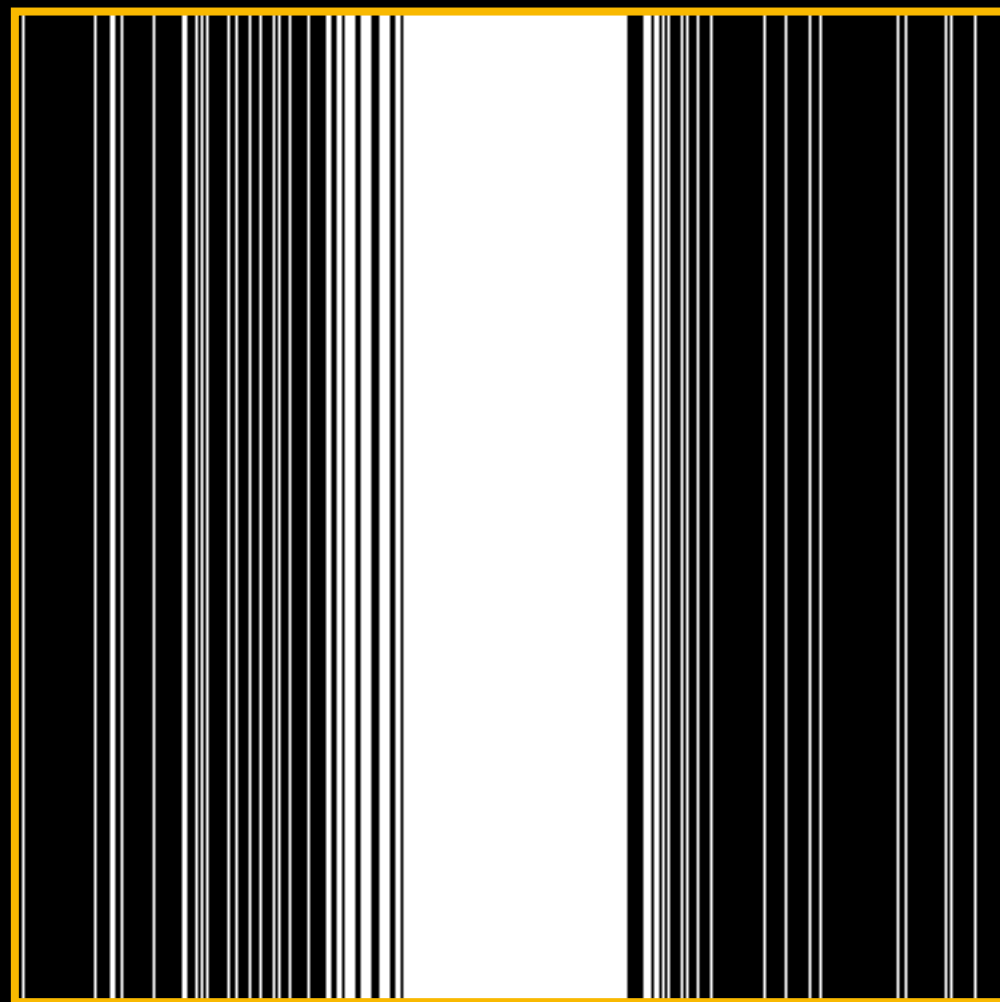
Coil sensitivity maps

Coil combined image

Multi-coil k-space data

Compressed sensing + Parallel imaging

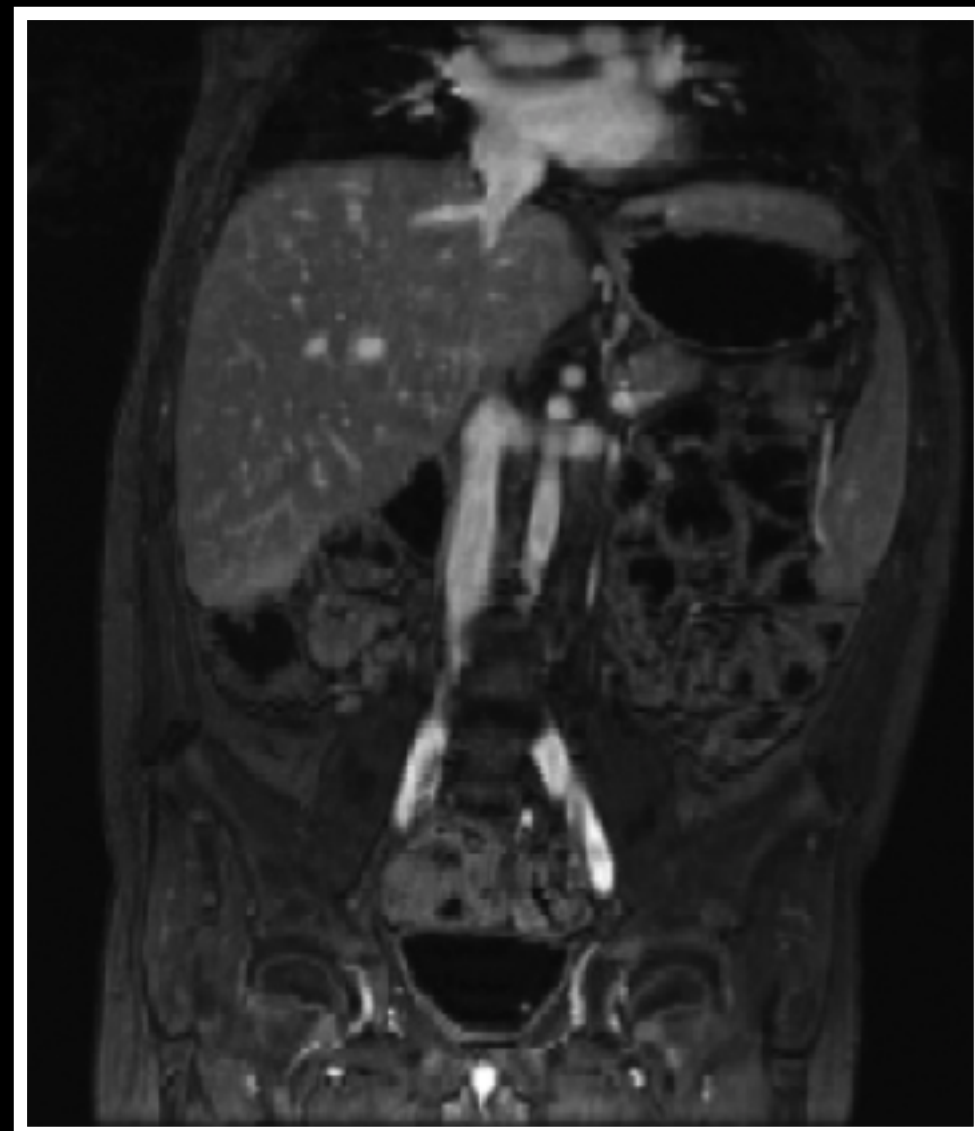
- Sampling trajectory:
 - The fully sampled region can be used to estimate coil sensitivity maps
 - The overall sampling scheme needs to generate incoherent artifacts



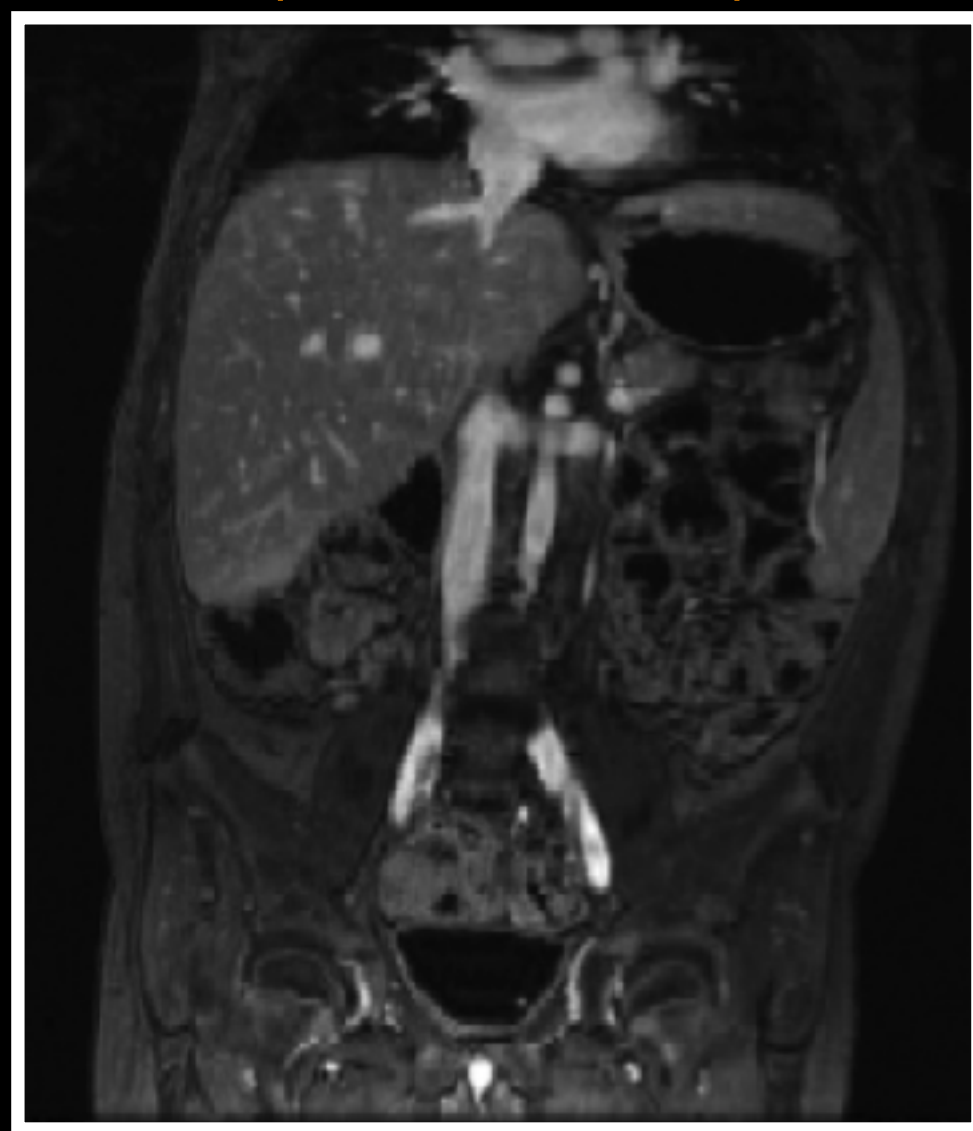
Coil compression

- A problem in applying compressed sensing reconstruction in some applications is the increased memory requirement and computational complexity due to a large number of coils.
- Coil compression (e.g., singular value decomposition-based technique) has been developed to reduce the number of coils before compressed sensing reconstruction.

Reference (32 coil elements)



Coil-compressed image
(6 virtual coils)



Error 20x



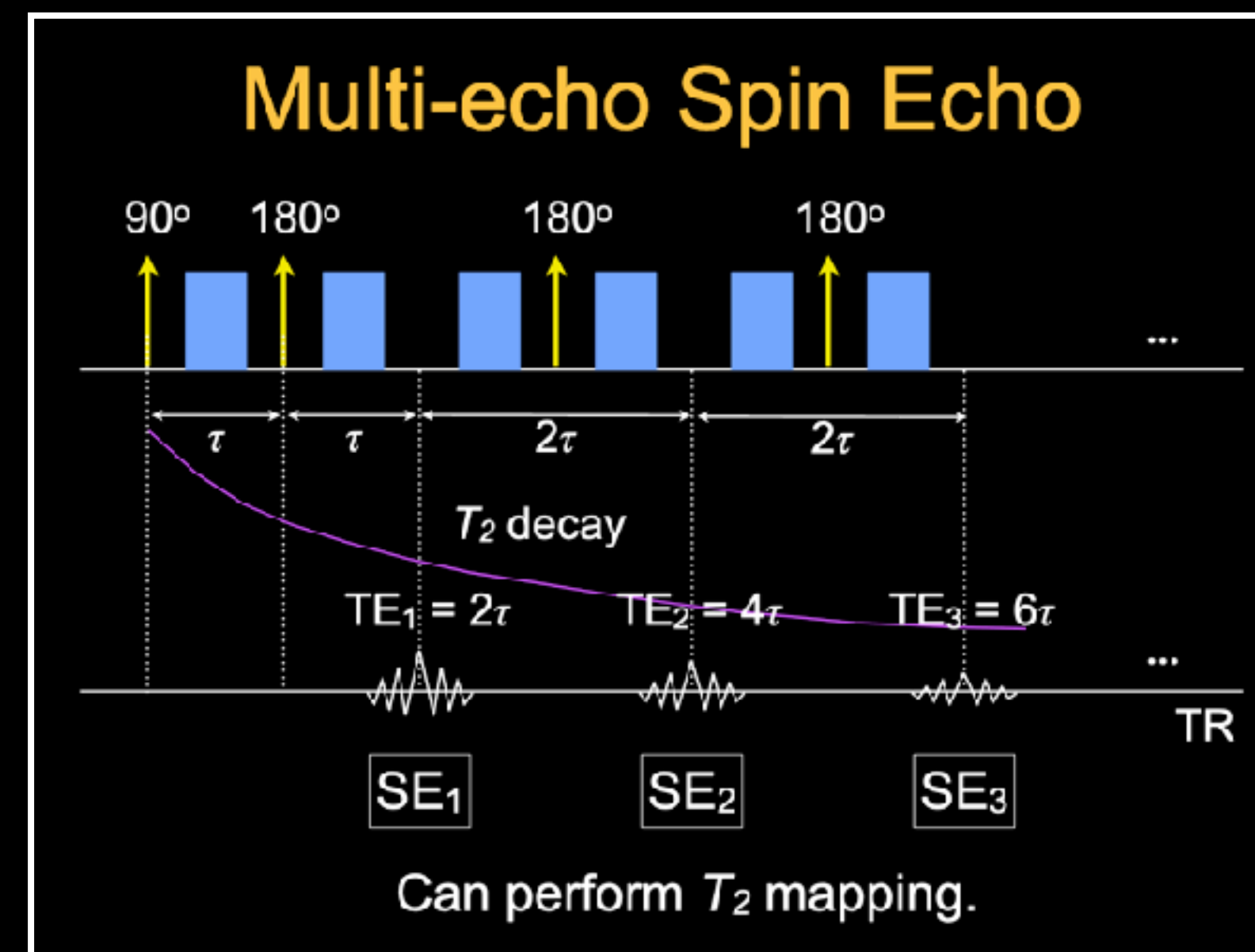
(Figures from: Zhang et al., MRM 2013)

Example (1): Knee T_2 mapping

- T_2 values in the knee cartilage have been used to detect disease- and treatment changes in articular cartilage.
- T_2 quantification in the knee cartilage can help depict early cartilage degeneration.
- Challenges: Conventional multi-echo spin echo-based sequences are slow



Multi-echo spin-echo images



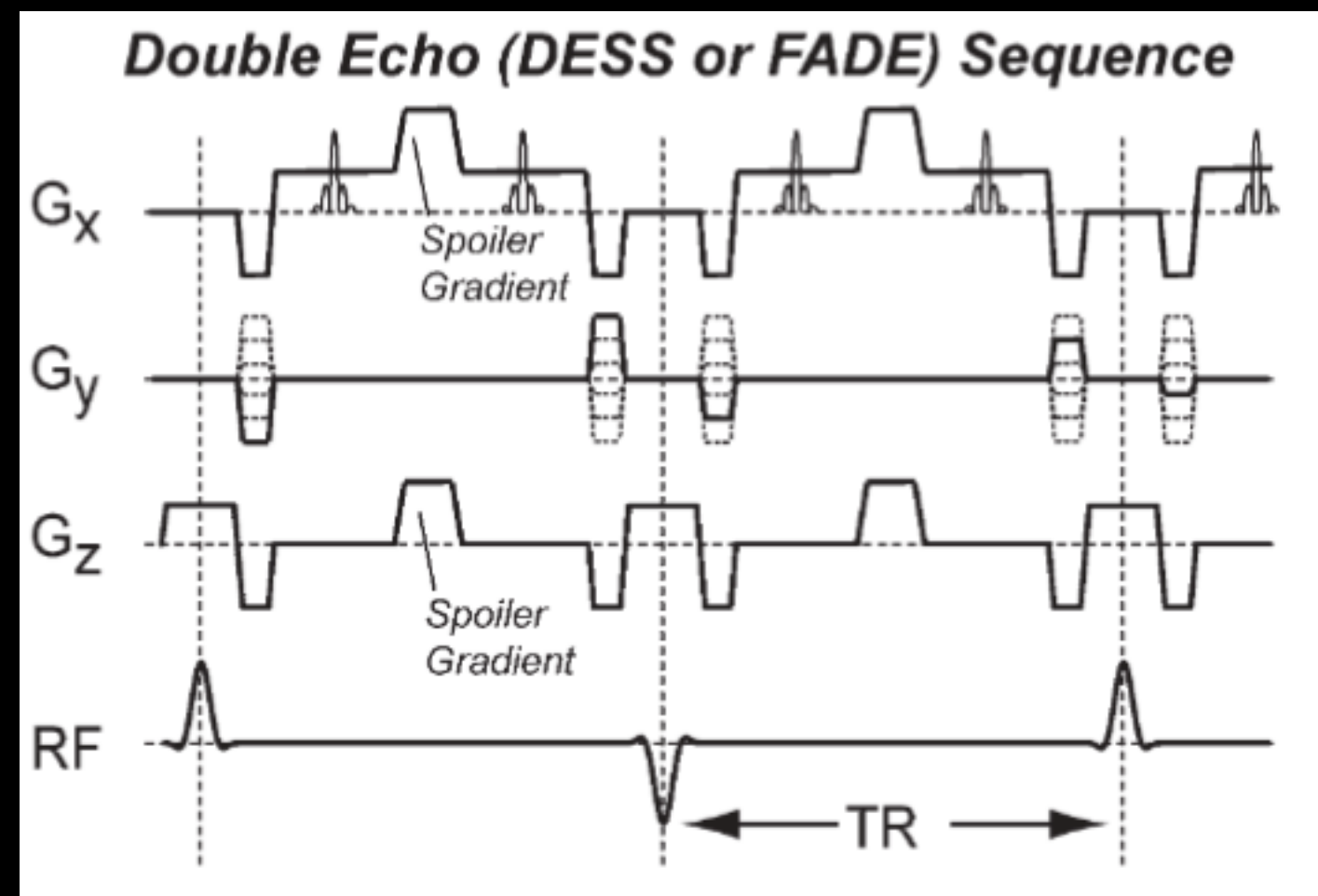
From previous lecture slide

Example (1): Knee T₂ mapping

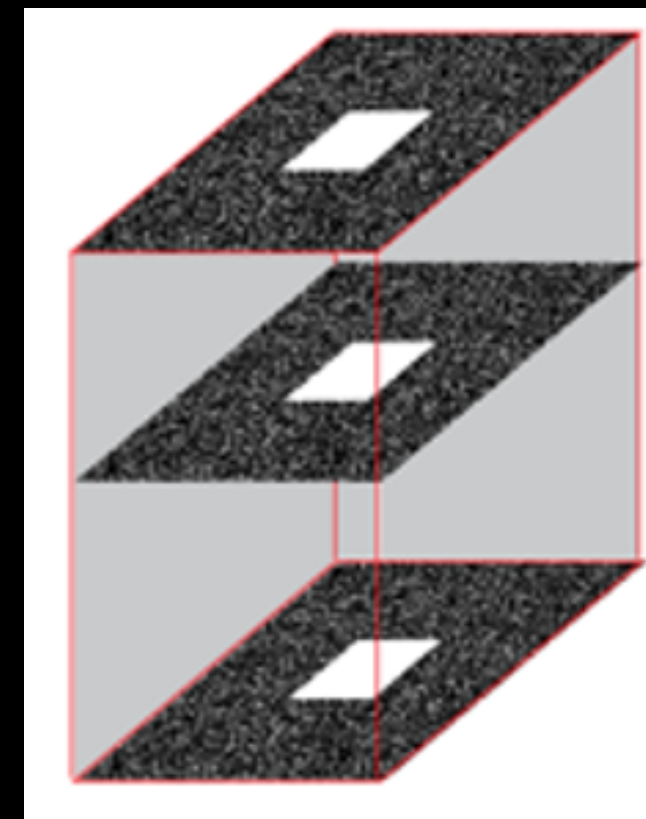
- Acceleration strategy
 - (1) Use a faster sequence: DESS (double/dual echo steady state)
 - (2) Use compressed sensing to accelerate

Variable density sampling

An extension to the gradient-spoiled GRE which acquires both SSFP-FID and SSFP-Echo



The difference between the two contrasts can be used to quantify T₂

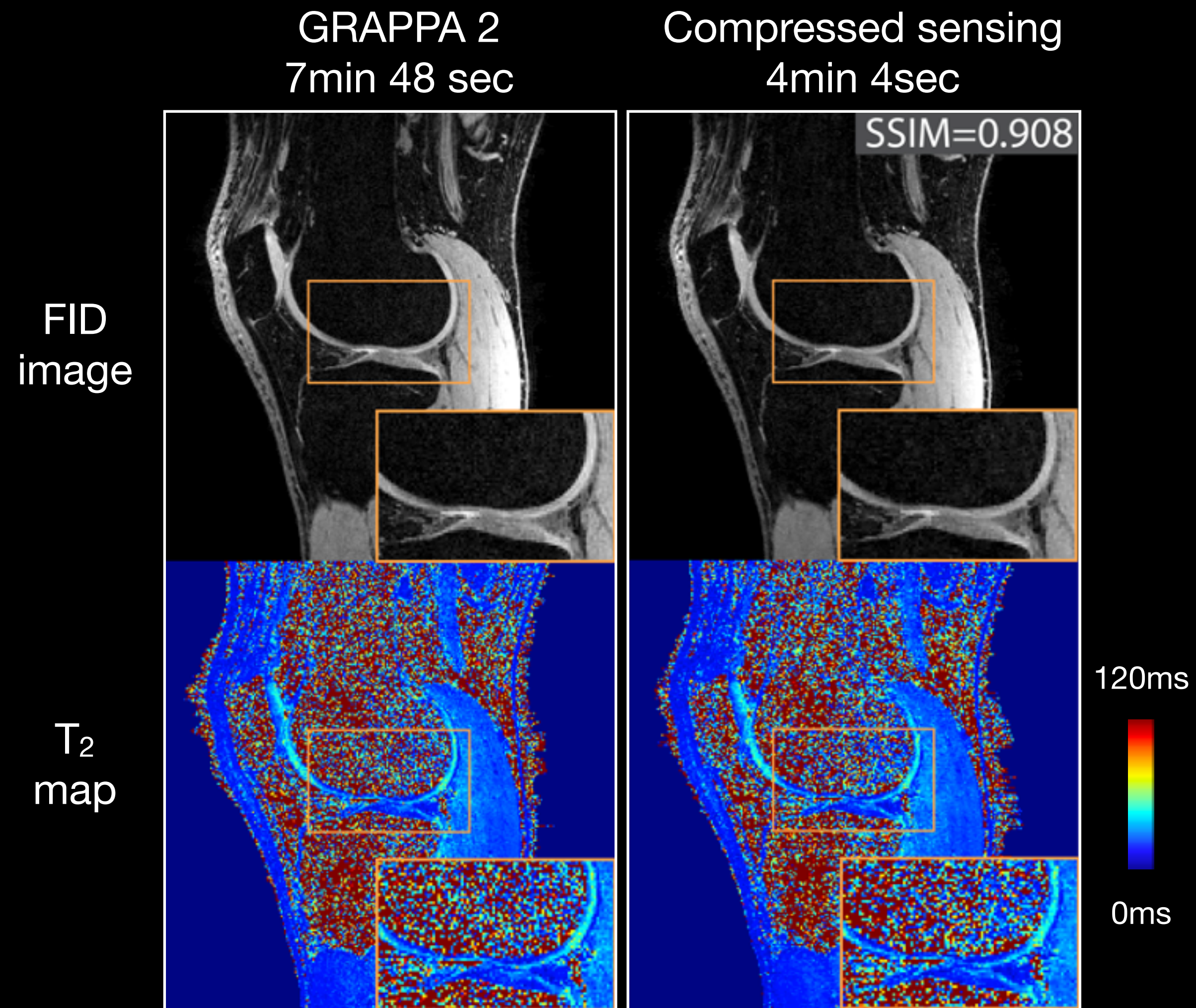


Cost function

$$\operatorname{argmin}_x \left(\| UFSx - y \|_2^2 + \lambda_1 \left(\| Wx_{fid} \|_1 + r \| Wx_{echo} \|_1 \right) + \lambda_2 \left(\| Dx_{fid} \|_1 + r \| Dx_{echo} \|_1 \right) \right)$$

U: k-space sampling pattern
 F: Fourier transform operator
 S: coil sensitivity maps
 x: reconstructed image
 y: acquired undersampled k-space data
 W: Wavelet transform operator
 D: total variation operator
 λ₁, λ₂: regularization parameters

Example (1): Knee T₂ mapping



(Figures from: Shih et al., ISMRM 2023)

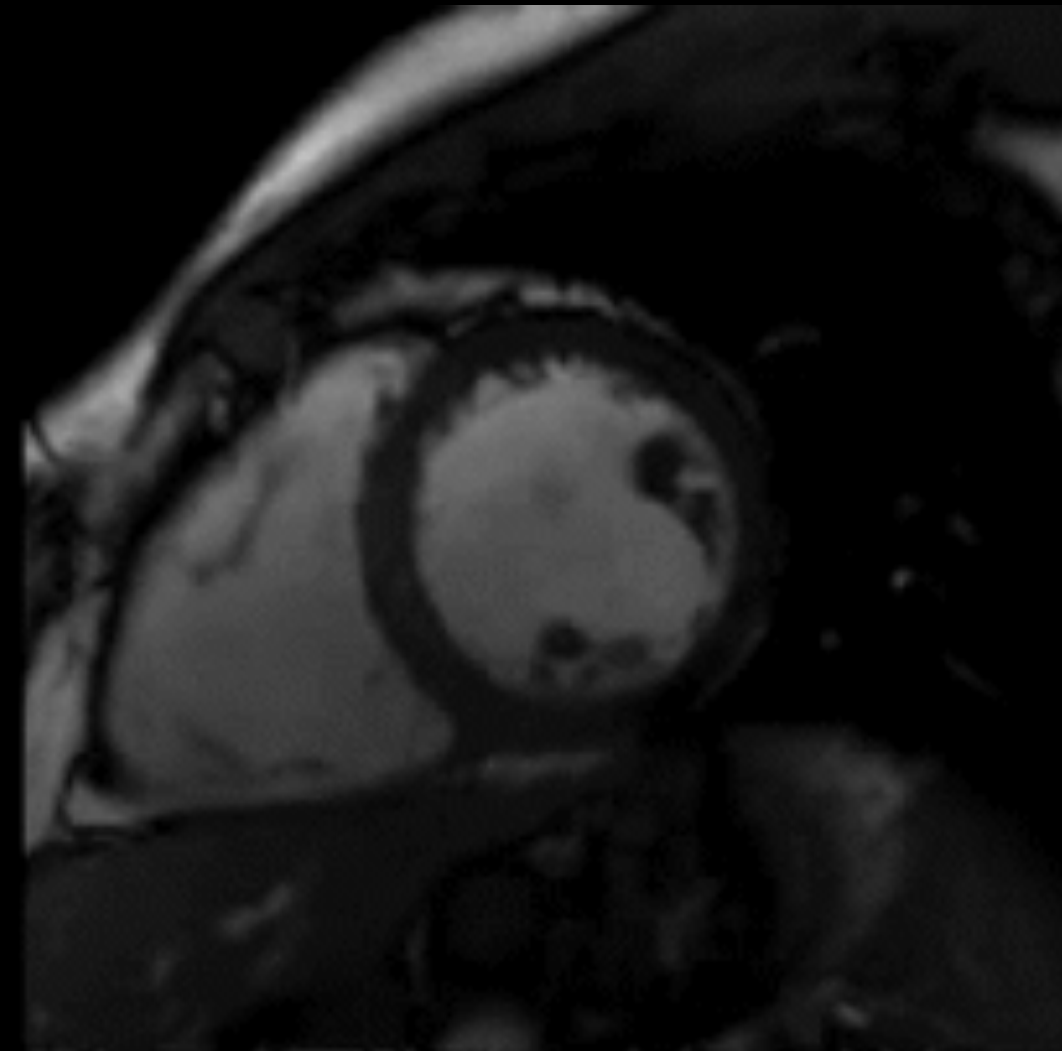
Example (1): Knee T₂ mapping

- Rapid knee cartilage T₂ mapping
 - Constraint: Sparsity in Wavelet transform and sparsity in total variation
 - Incoherent measurement: variable density random sampling
 - Optimization function: $\operatorname{argmin}_x \left(\| UFSx - y \|_2^2 + \lambda_1 (\| Wx_{fid} \|_1 + r \| Wx_{echo} \|_1) + \lambda_2 (\| Dx_{fid} \|_1 + r \| Dx_{echo} \|_1) \right)$
 - Reconstruction: non-linear conjugate gradient method

U: k-space sampling pattern
F: Fourier transform operator
S: coil sensitivity maps
x: reconstructed image
y: acquired undersampled k-space data
W: Wavelet transform operator
D: total variation operator
 λ_1, λ_2 : regularization parameters

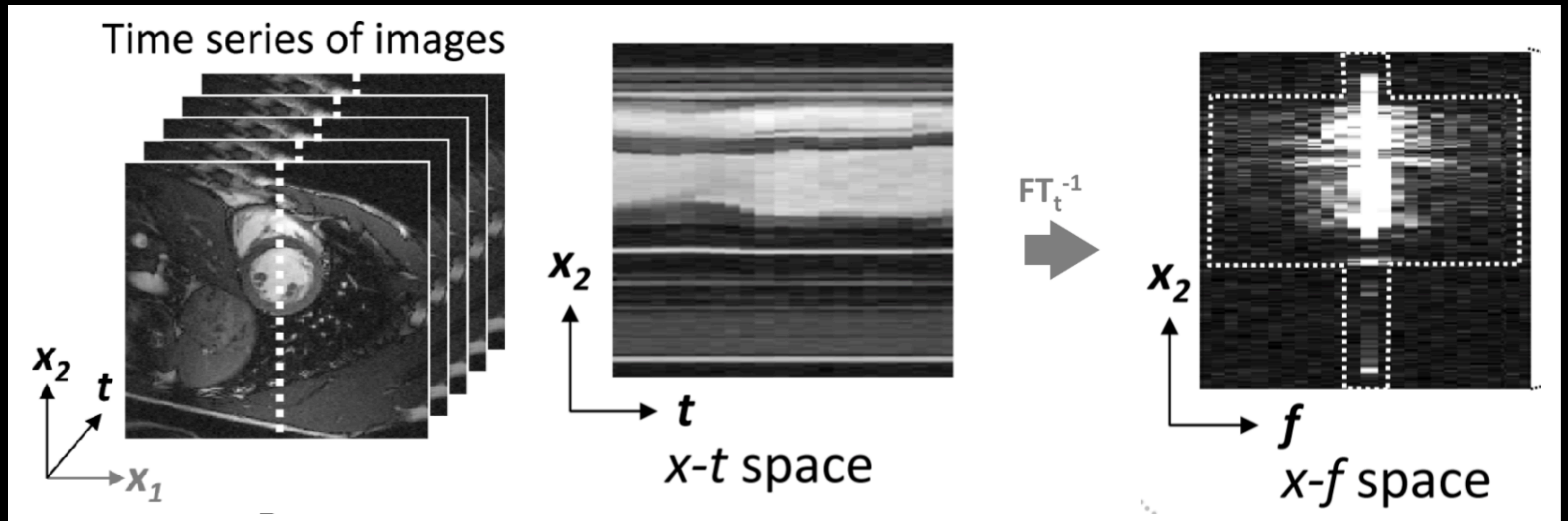
Example (2): Cardiac cine imaging

- Cardiac cine imaging for information of the heart function throughout the cardiac cycle
- Challenges: accelerating data acquisition without compromising the high resolution and image quality requirements



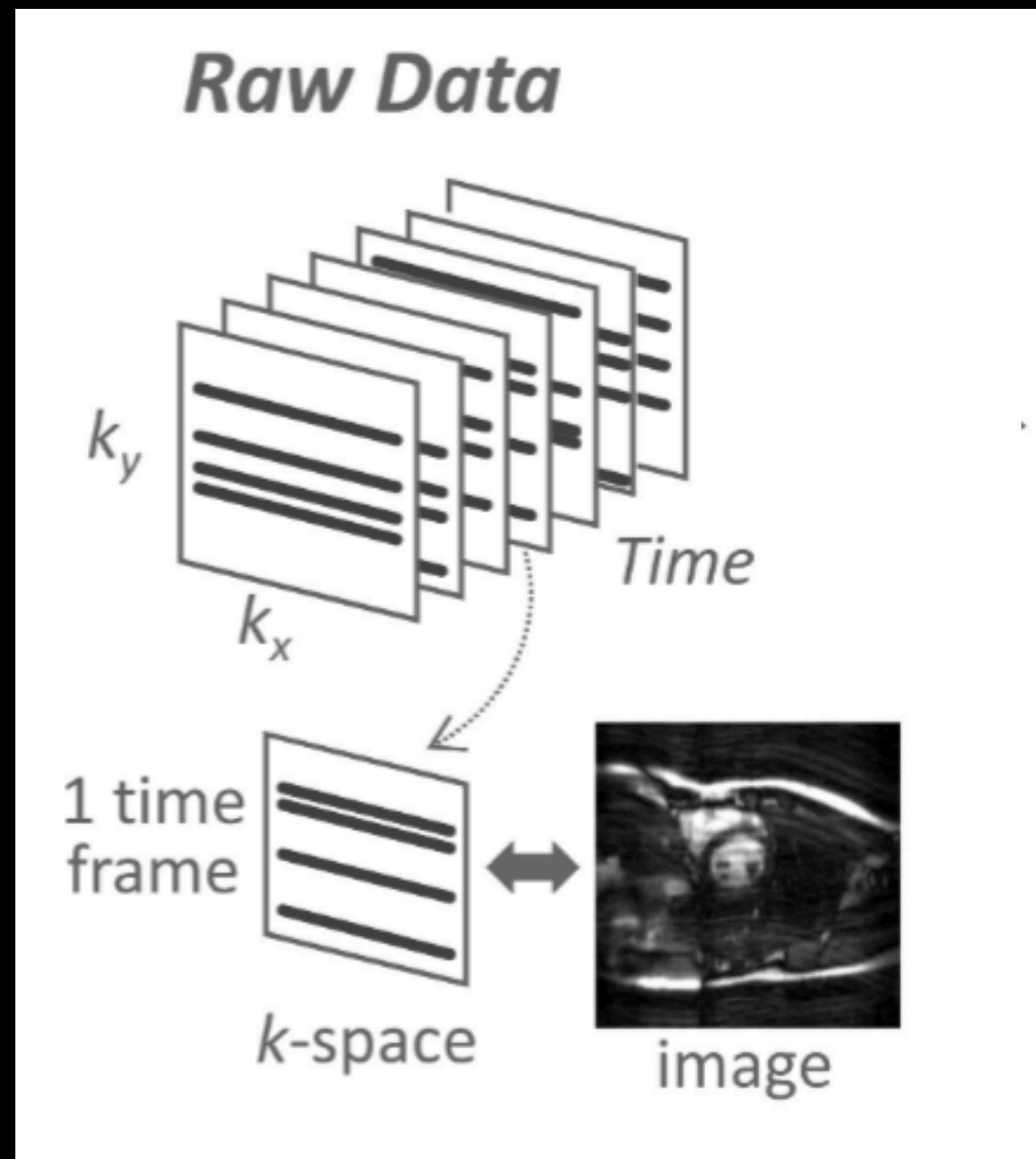
Example (2): Cardiac cine imaging

- Sparsity in the x - f space

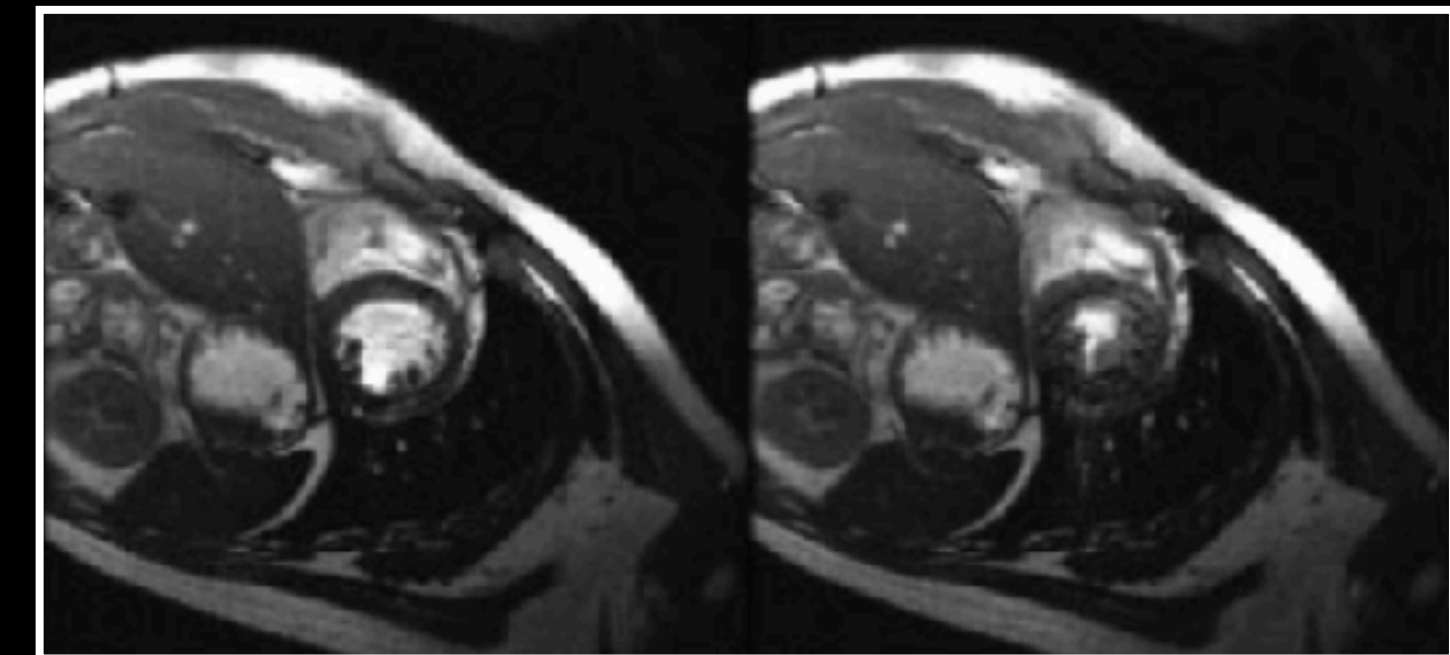


Example (2): Cardiac cine imaging

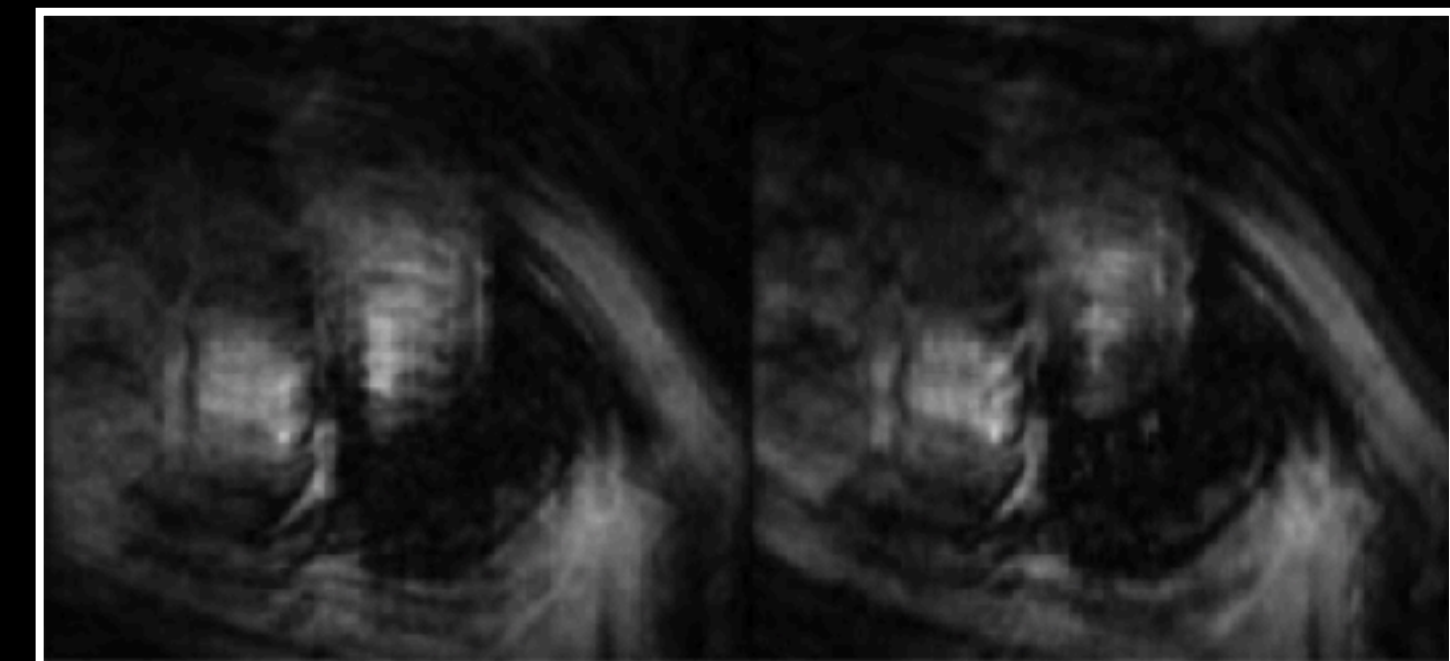
k-t sampling pattern



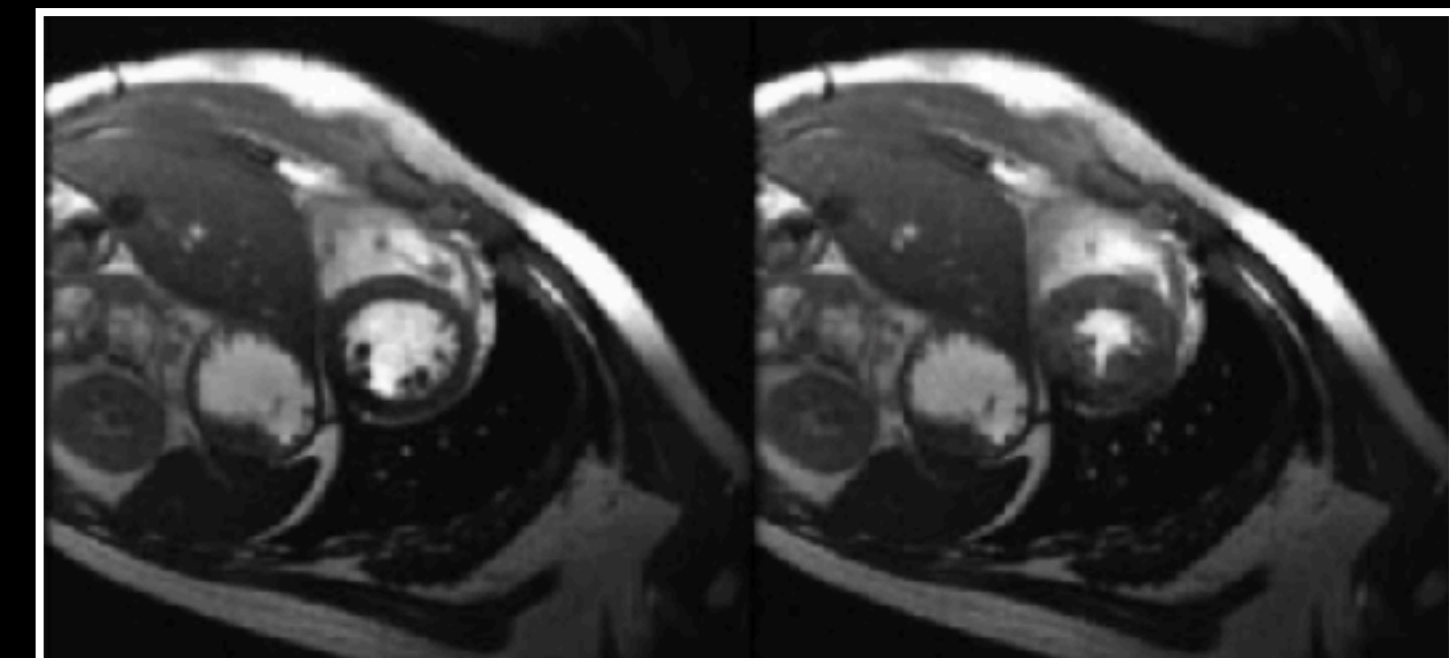
Fully sampled



**6x acceleration
with zero-padding**



k-t FOCUSS results



Example (2): Cardiac cine imaging

- k-t FOCUSS¹ (*k-t FOCal Underdetermined System Solver*)

- Application: cardiac cine imaging

- Constraint: sparsity in the x-f space

- Incoherent measurement: k-t undersampling

- Optimization function: $\min_{\rho} \|y - DFS\rho\|_2^2 + \lambda \|\rho\|_1$

Let $\rho = \rho_0 + \Delta\rho$

$\min_{\rho} \|y - DFS(\rho_0 + \Delta\rho)\|_2^2 + \lambda \|\Delta\rho\|_1$

- Reconstruction: reweighted quadratic optimization

y: acquired k-space data
D: k-t sampling pattern
F: Transform operator between
k-space and x-f space
S: coil sensitivity maps
 ρ : reconstructed x-f space
 λ : regularization parameter

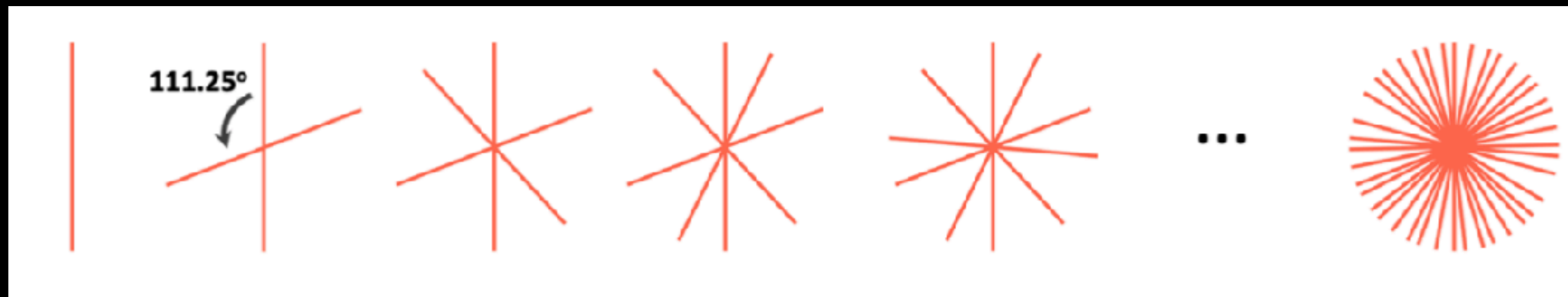
Example (3): Free-breathing radial MRI

- Radial MRI with inherent motion robustness can be used for free-breathing MRI
- Radial undersampling results in incoherent artifacts

Linear radial MRI

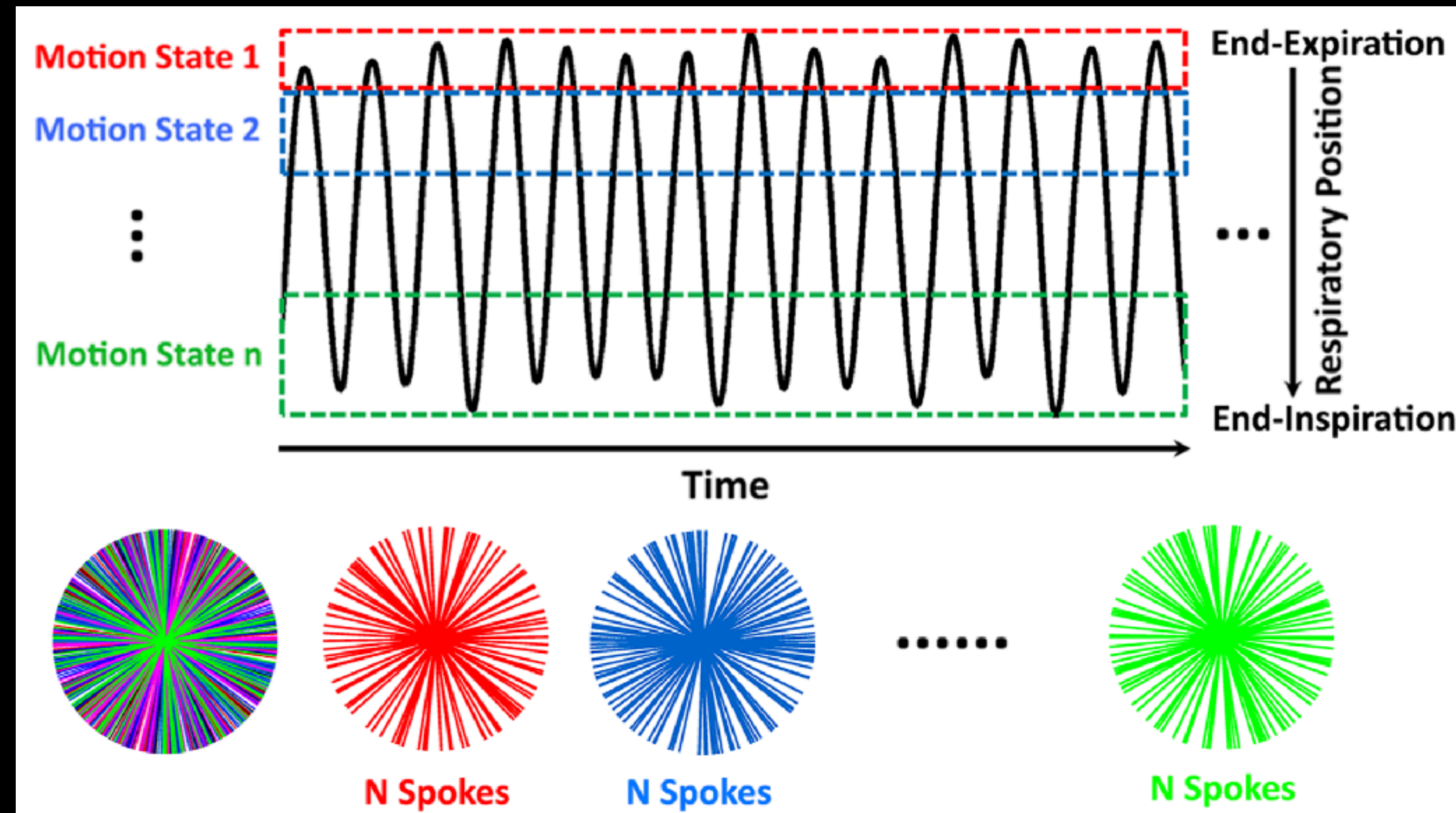


Golden-angle radial MRI

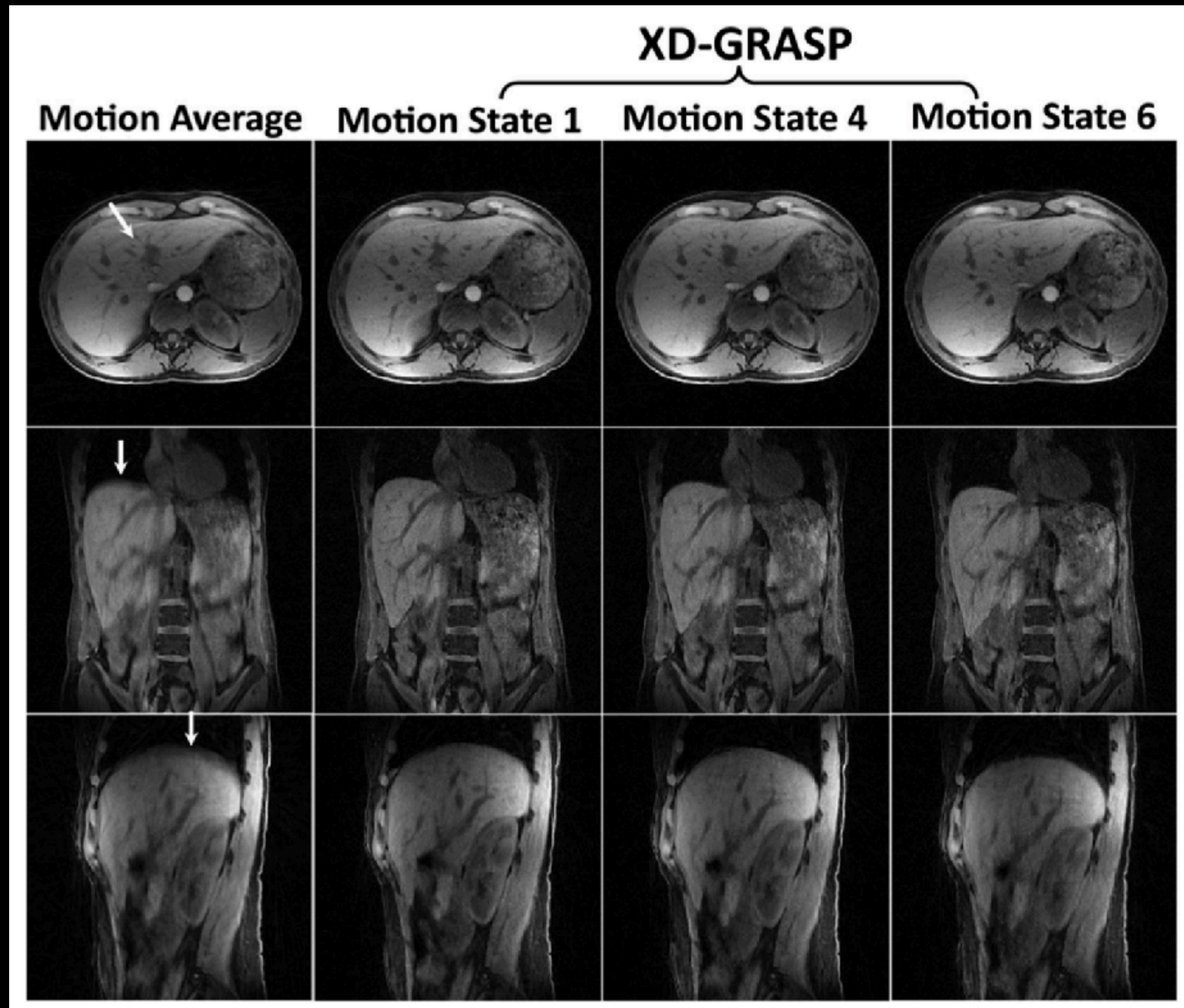


Example (3): Free-breathing radial MRI

- Stack-of-radial MRI provides self-navigation to track breathing motion
- We can group the k-space data into different motion states



Example (3): Free-breathing radial MRI



(Figure from: Feng et al., MRM 2016)

Example (3): Free-breathing radial MRI

- XD-GRASP¹ (*Golden-angle radial MRI with reconstruction of extra motion-state dimensions using compressed sensing*)
 - Application: free-breathing abdominal imaging
 - Constraint: temporal finite differences (or total variation) in dynamic dimension
 - Incoherent measurement: undersampled golden-angle radial MRI
 - Optimization function: $\min_x \left\| FCx - y \right\|_2^2 + \lambda_1 \left\| S_1x \right\|_1 + \lambda_2 \left\| S_2x \right\|_1$
 - Reconstruction: non-linear conjugate gradient

Compressed sensing MRI

- Limitations:
 - Requiring high computational complexity to solve the nonlinear reconstruction problem
 - Reconstruction result is dependent on the choice of regularization parameters
- Other related constrained reconstruction methods
 - Dictionary-based compressed sensing MRI
 - MRI reconstruction using low-rank constraints
 - ...

Take home message

- 3 main components in compressed sensing MRI
 - The image has a **sparse representation** in some transform domain
 - The k-space sampling trajectory generates **incoherent artifacts** in the sparse transform domain
 - It involves a **nonlinear reconstruction** method

Take home message

- If we want apply compressed sensing to accelerate an application, check:
- (1) Can the images be sparsified in a certain (transform) domain?
 - Wavelet transform
 - Spatial total variation in images
 - Total variation in temporal frames
 - x-f space
 - ...
- (2) Can the sampling pattern generate incoherent artifacts?
 - Variable density sampling pattern
 - Radial acquisition
 - Spiral acquisition
 - ...

Thanks!

- Next time
 - Deep learning MRI reconstruction

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