MRI Signal Equation, Basic Image Reconstruction

M219 Principles and Applications of MRI

Holden H. Wu, Ph.D.

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UCLA
Department of Radiological Sciences
David Geffen School of Medicine at UCLA

Outline

- MRI Signal Equation (review)
- Basic Image Reconstruction
- Sampling Considerations
- Noise Considerations
- Reconstruction Considerations
  - Zero padding (interpolation)
  - Windowed recon to reduce Gibb's ringing
  - Multi-channel (coil) reconstruction

MRI Signal Equation

$$s\left(t\right) = \int\int_{x,y} \vec{M}_{xy}^{0}\left(\vec{r}\right) \cdot e^{-i\Delta\omega(\vec{r})t} \mathrm{d}\vec{r}$$

The MRI Signal Equation is the…

$$s\left(t\right) = \int\int_{x,y} \vec{M}_{xy}^{0}\left(x, y\right) \cdot e^{-i\Delta\omega(x,y)t} \mathrm{d}x\mathrm{d}y$$

…2D Fourier Transform!

$$\Delta\omega(x, y) = \gamma G_x \cdot x + \gamma G_y \cdot y$$

Gradients define Δw

$$k_x(t) = \frac{\gamma}{2\pi} G_x t \qquad k_y(t) = \frac{\gamma}{2\pi} G_y t$$

k-space is convenient…

$$s\left(k_x(t), k_y(t)\right) = \int\int_{x,y} \underbrace{\vec{M}_{xy}^{0}\left(x, y\right)}_{I\left(\vec{r}\right)} \cdot e^{-i2\pi[k_x(t)x + k_y(t)y]} \mathrm{d}x\mathrm{d}y$$

Image Reconstruction

$$S = \mathcal{T}\left\{I\right\}$$

Data Consistency
Constraint

Measured
Signal

Spatial Information
Encoding Scheme
(Fourier Transform)

Image
Function

$$I = \mathcal{T}^{-1}\left\{S\right\}$$

The Fourier Transform

$$S(\vec{k}) = \int_{-\infty}^{+\infty} I\left(\vec{r}\right) e^{-i2\pi \vec{k}\cdot\vec{r}} d\vec{r}$$

MRI Signal Equation

$$S(\vec{k}) \xleftrightarrow{\mathcal{F}} I(\vec{r})$$

$$S(k_x) = \int_{-\infty}^{+\infty} I\left(x\right) e^{-i2\pi(k_x x)} dx$$ 1D

$$S(k_x, k_y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} I\left(x, y\right) e^{-i2\pi(k_x x + k_y y)} dxdy$$ 2D

$$S(k_x, k_y, k_z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} I\left(x, y, z\right) e^{-i2\pi(k_x x + k_y y + k_z z)} dxdydz$$ 3D

Image Reconstruction

Given $S(\vec{k}_n) = \int_{-\infty}^{+\infty} I\left(\vec{r}\right) e^{-i2\pi \vec{k}_n \cdot \vec{r}} d\vec{r}$ MRI Signal Equation

How do we determine $I\left(\vec{r}\right)$?

Image Reconstruction

$$S(\vec{k}_n) = \int_{-\infty}^{+\infty} I(\vec{r})\, e^{-i2\pi \vec{k}_n \cdot \vec{r}} d\vec{r}$$ MRI Signal Equation

$$\mathcal{D} = \left\{ \vec{k}_n = n\Delta\vec{k}, n = \ldots, -2, -1, 0, 1, 2, \ldots \right\}$$

Uniform *k*-space sampling

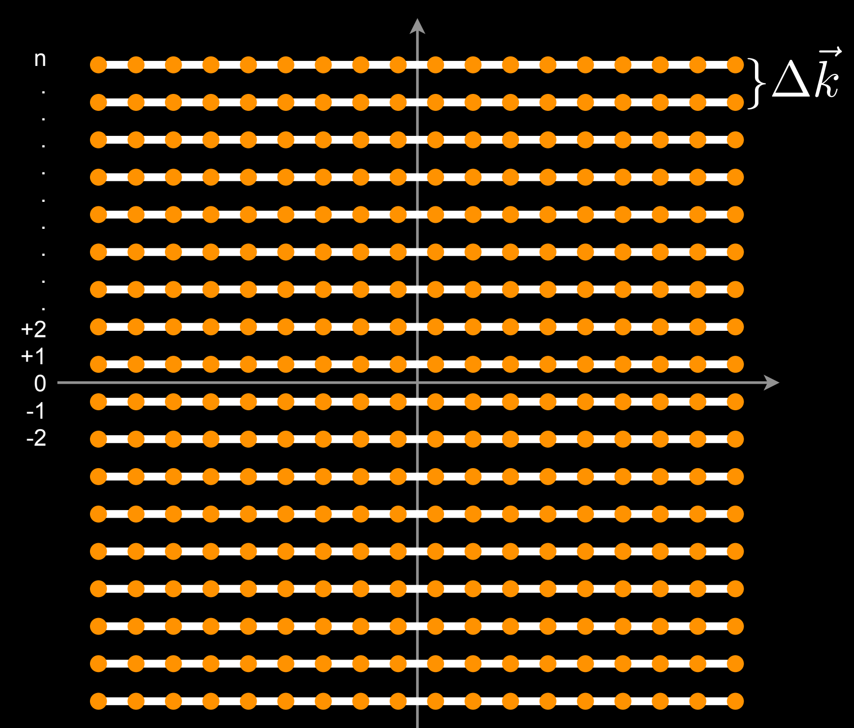


Image Reconstruction

$$S(\vec{k}_n) = \int_{-\infty}^{+\infty} I\left(\vec{r}\right) e^{-i2\pi\vec{k}_n\cdot\vec{r}} d\vec{r}$$

$$\Downarrow$$

$$\mathcal{D} = \left\{ \vec{k}_n = n\Delta\vec{k}, n = \ldots, -2, -1, 0, 1, 2, \ldots \right\}$$

Uniform *k*-space sampling

$$\Downarrow$$

$$S\left[n\right] = S\left(n\Delta k_x\right) = \int_{-\infty}^{+\infty} I\left(x\right) e^{-i2\pi n\Delta k_x\cdot x} dx$$

One-dimensional Case

Image Reconstruction

$$S[n] = \underbrace{S(n\Delta k_x)} = \int_{-\infty}^{+\infty} \underbrace{I(x)}\, e^{-i2\pi n\Delta k_x \cdot x} dx$$

This is what we measure!

This is what we want!

Image Reconstruction

$$S[n] = \underbrace{S(n\Delta k_x)} = \int_{-\infty}^{+\infty} \underbrace{I(x)} e^{-i2\pi n\Delta k_x \cdot x} dx$$

Eqn. 6.9

This is what we measure! This is what we want!

We can show the following...(Page 191 in Lauterbur).

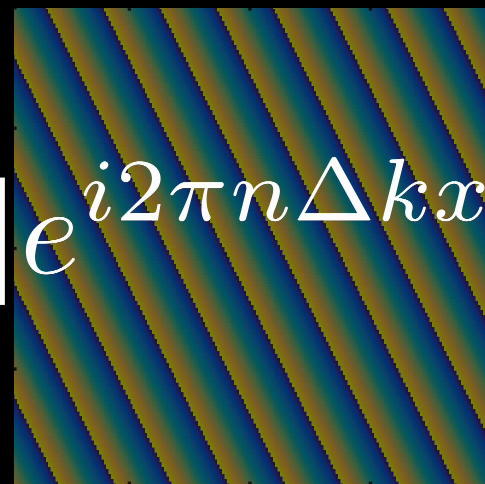
$$\sum_{n=-\infty}^{\infty} S[n] e^{i2\pi n\Delta k x} = \frac{1}{\Delta k} \sum_{n=-\infty}^{\infty} I\left(x - \frac{n}{\Delta k}\right)$$

Eqn. 6.10

Fourier Series

Periodic Extension of I(x)

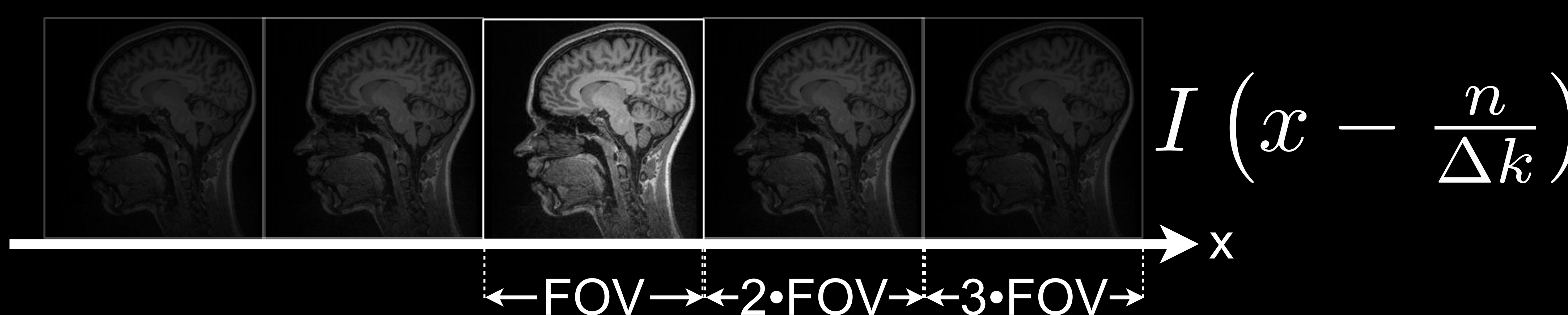
Image Reconstruction



$$\sum_{n=-\infty}^{\infty} S[n] e^{i2\pi n\Delta k x} = \frac{1}{\Delta k} \sum_{n=-\infty}^{\infty} I\left(x - \frac{n}{\Delta k}\right)$$

- Fourier series
- Δk is the fundamental frequency
- S[n] coefficient of the nth harmonic

- Periodic extension of $I(x)$
- n is an integer
- Period is 1/Δk=FOV



Periodic extensions of a object/function.

Sampling Considerations

Infinite Sampling

$S(k)$ is measured at $k \in \mathcal{D}$

$$\mathcal{D} = \{n\Delta k, -\infty < n < +\infty\}$$

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Can *I(x)* be recovered from its periodic extension?

$$\sum_{n=-\infty}^{\infty} S[n] e^{i2\pi n\Delta k x} = \frac{1}{\Delta k} \sum_{n=-\infty}^{\infty} I\left(x - \frac{n}{\Delta k}\right)$$

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If $I(x) = 0$ on $|x| > FOV_x/2 \left(i.e.\ \Delta k < \frac{1}{FOV_x}\right)$, then

Infinite Sampling

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If $I(x) = 0$ on $|x| > FOV_x/2 \left(i.e.\ \Delta k < \frac{1}{FOV_x}\right)$, then

$$I(x) = \Delta k \sum_{n=-\infty}^{\infty} S[n] e^{i2\pi n\Delta k x}, \ |x| < \frac{1}{\Delta k}$$ Eqn. 6.16

But ∞ takes forever...

Finite Sampling

$S(k)$ is measured at $k \in \mathcal{D}$

$$\mathcal{D} = \{n\Delta k, -N/2 \leq n \leq +N/2\}$$

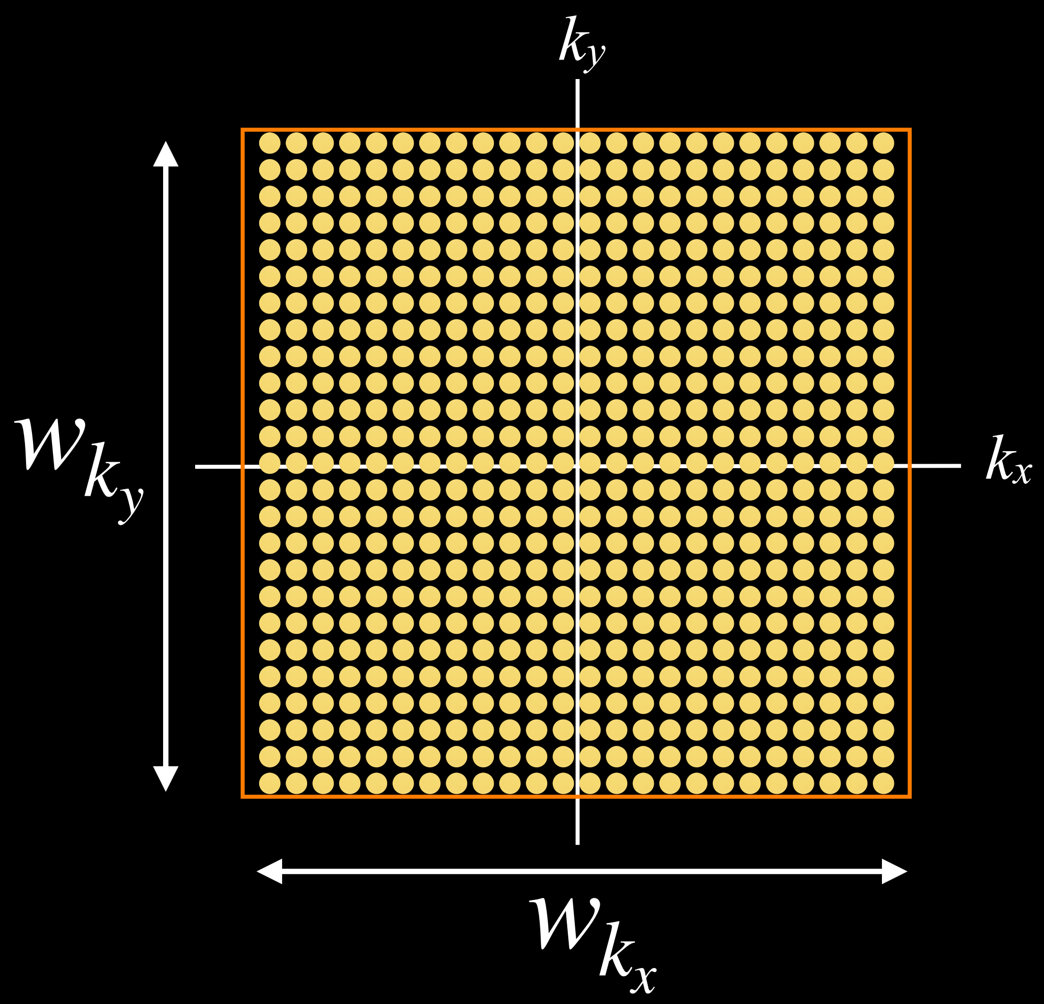
Fourier Step-size

Number of Sample Points

$$I(x) = \Delta k \sum_{n=-N/2}^{N/2-1} S[n] e^{i2\pi n \Delta k x}, \; |x| < \frac{1}{\Delta k}$$ Eqn. 6.20

This is the fundamental image reconstruction equation for MRI.

Sampling Considerations



$$\Delta k_x = \frac{1}{FOV_x} \qquad \Delta k_y = \frac{1}{FOV_y}$$

$$w_{k_x} = \frac{1}{\Delta x} \qquad w_{k_y} = \frac{1}{\Delta y}$$

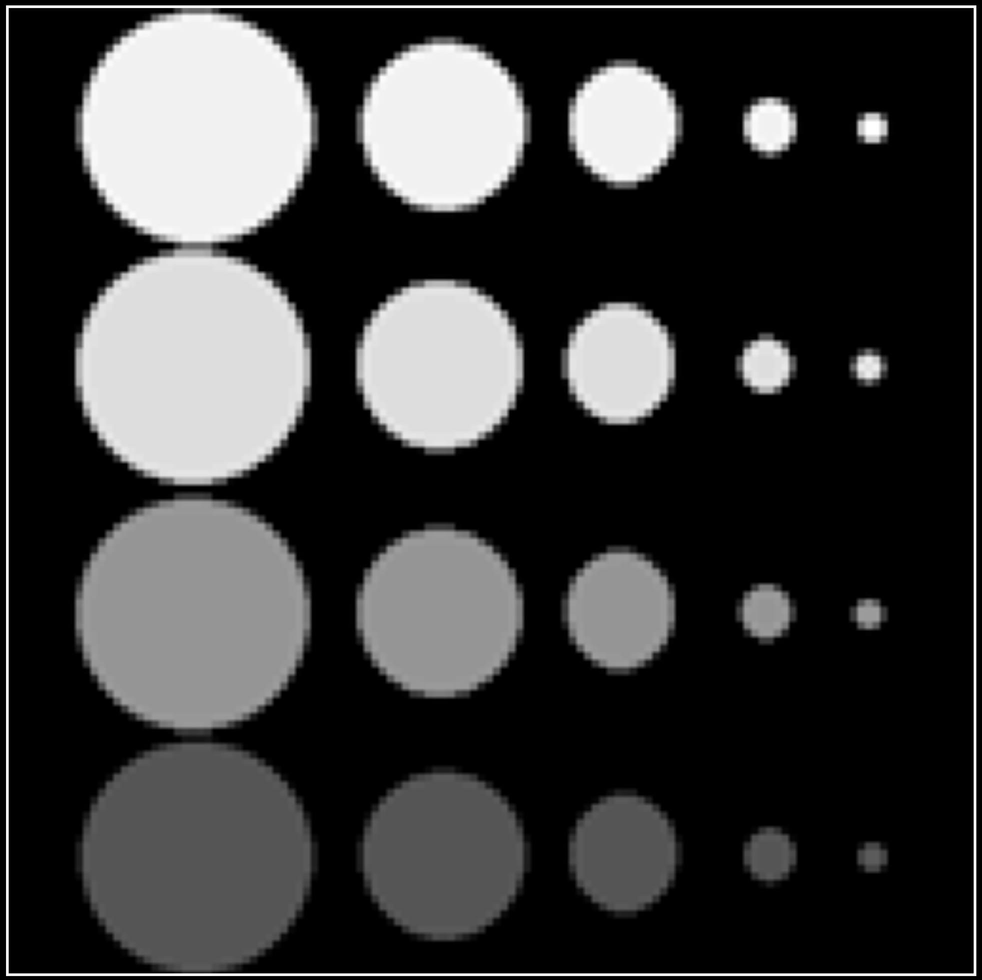
Review Sampling Theorem

Review Lectures 9&10 on Spatial Localization

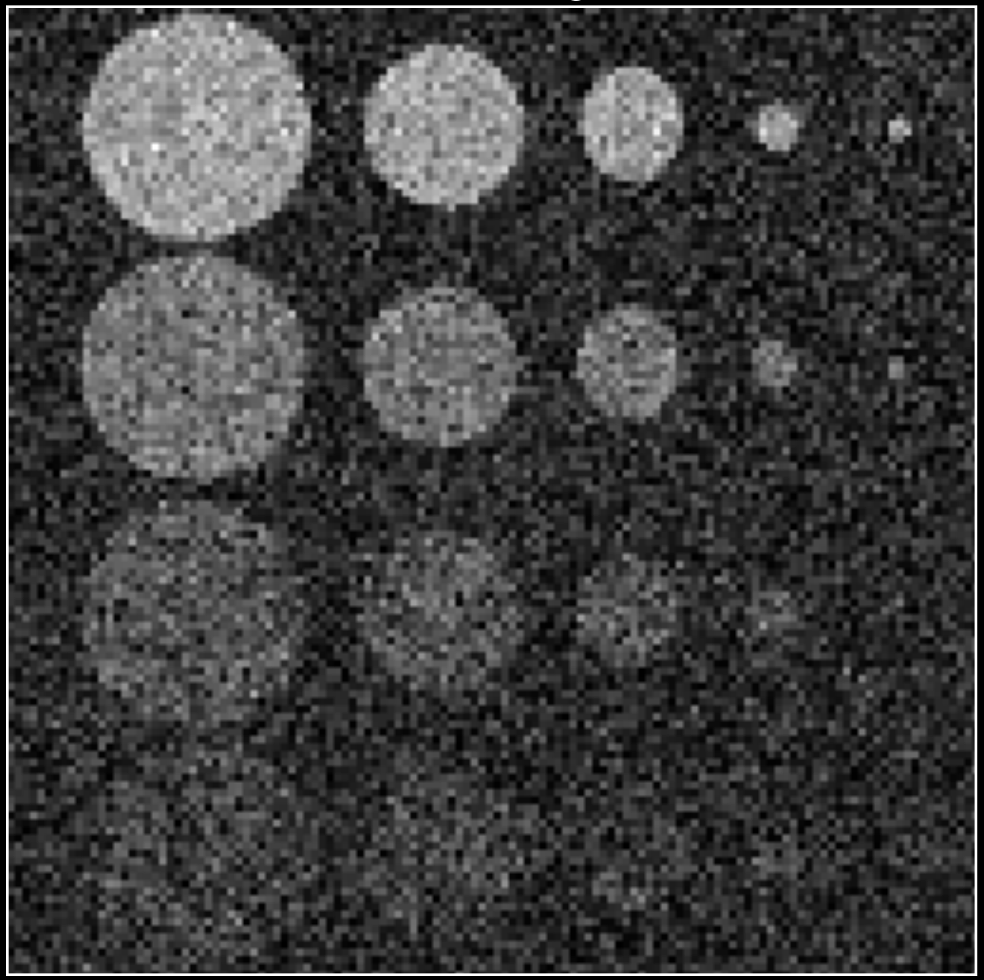
Noise Considerations

Noise Considerations

Noise Free

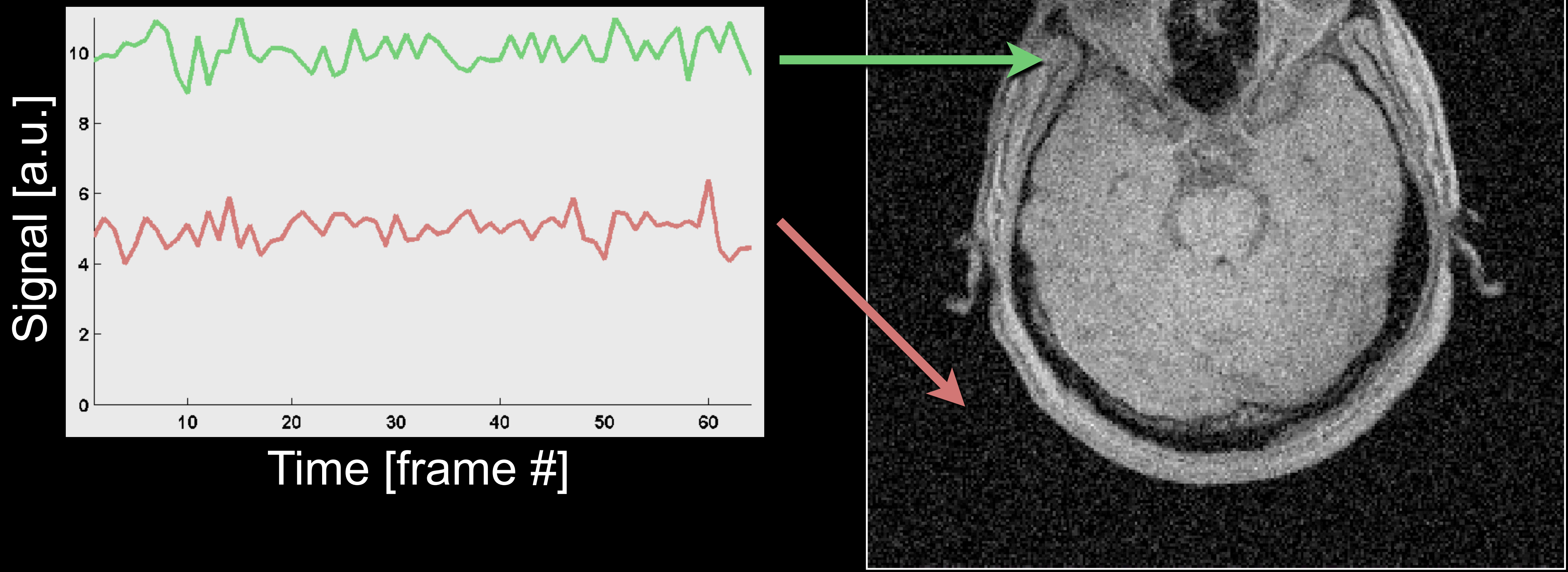


Noisy



Noise Considerations

- Signal-to-Noise Ratio (SNR)



Noise Considerations

- Signal-to-Noise Ratio (SNR)
  - A fundamental measure of image quality
  - $SNR \triangleq \dfrac{signal\ amplitude}{\sigma\ of\ noise}$
  - $SNR_{dB} = 20 \cdot log(SNR)$

Nishimura Ch. 7.5

Noise Considerations

- Noise Sources
  - Thermal (Brownian motion of electrons)
  - Coil resistance, sample (body) resistance

  - Power spectral density:
    $N(f) = 4kTR$ and $N(\Delta f) = 4kTR \cdot \Delta f$
  - Modeled as additive white Gaussian (AWG) noise

  - Noise from the body typically dominates,
    $SNR \propto B_0$

Nishimura Ch. 7.5

Noise Considerations

- Image Noise Statistics
  - Physical real-valued signal
    $\xi_p(t) = s_p(t) + n_p(t)$
  - Sampled (Nyquist) demodulated complex signal
    $\hat{\xi}(j) = \hat{s}(j) + \hat{n}(j)$
  - $\hat{n}$ is bivariate (complex) zero-mean Gaussian, with real/imag components each with σ_n^2

Nishimura Ch. 7.5

Noise Considerations

- Image Noise Statistics
  - 2D Cartesian *k*-space sampling is uniform and 2D FT is unitary, thus noise in the image domain will also be AWG
  - The magnitude operation $|I(a,b)|$ alters noise statistics
  - Background (*I* is zero-mean): Rayleigh distr.
  - Signal regions: Rician distr.

Nishimura Ch. 7.5

Noise Considerations

- Effect of Acquisition Time
  - Simple 1D example (impulse in image space)
  - *N* samples in *k*-space, each with amplitude *A*
  - Noise variances add (independence)
  - $$SNR = \frac{\sum_{j=1}^{N} A}{\sqrt{\sum_{j=1}^{N} \sigma_n^2}} = \frac{NA}{\sqrt{N\sigma_n^2}} = \frac{\sqrt{N}A}{\sigma_n}$$

Nishimura Ch. 7.5

Noise Considerations

- Effect of Signal Averaging
  - Average separate measurements of the same *k*-space data samples (e.g., 2 measurements)
  - Signal amplitudes add
  - Noise variances also add (independence)
  - $$SNR_{2Ave} = \frac{\sum_{j=1}^{N} 2A}{\sqrt{\sum_{j=1}^{N} 2\sigma_n^2}} = \frac{2NA}{\sqrt{2N\sigma_n^2}} = \frac{\sqrt{2NA}}{\sigma_n}$$
  - $$SNR_{2Ave} = \sqrt{2} \cdot SNR$$

Nishimura Ch. 7.5

Noise Considerations

- Effect of Readout Time
  - Double readout duration T_{read}
  - Typically, also double sampling interval Δt to maintain k-space sampling extent
  - $\Delta f \propto 1/(\Delta t)$: halves the signal bandwidth Δf
  - Recall that $\sigma_n^2 \propto \Delta f$
  - $$SNR_{2\cdot Tread} = \frac{NA}{\sqrt{N\sigma_n^2/2}} = \frac{\sqrt{2NA}}{\sigma_n}$$
  - $SNR_{2\cdot Tread} = \sqrt{2} \cdot SNR$

Nishimura Ch. 7.5

Noise Considerations

- Summary of Acquisition Time Effects
  - $SNR \propto \sqrt{N_{ave} \cdot T_{read}}$
  - $SNR \propto \sqrt{measurement\ \ time}$
- Effect of Spatial Resolution
  - $SNR \propto (\delta_x)(\delta_y)(\delta_z)$
- Other factors
  - $SNR \propto f(\rho, T_1, T_2, \ldots)$

Nishimura Ch. 7.5

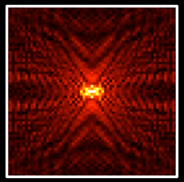
Zero Padding

Zero-Padding

- Append zeros to *k*-space data before FFT
  - Append symmetrically about *k*-space
- Why?
  - If $N=2^n$, then the radix-2 FFT can be used
  - Increases the “digital” resolution; interpolates pixels in image space
  - Reconstruction with correct aspect ratio
  - Starting point for iterative reconstructions; or a reference for comparisons

Asymmetric Resolution

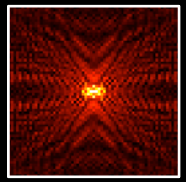
Low-Res Data



64x64

Asymmetric Resolution

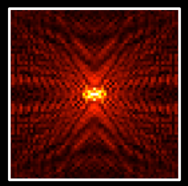
Low-Res Data



64x64

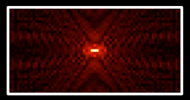
Asymmetric Resolution

Low-Res Data

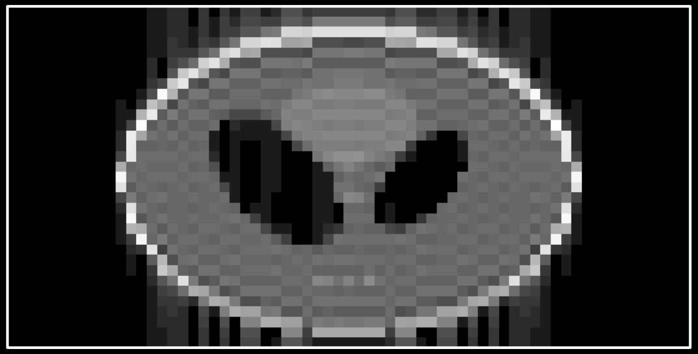


64x64

Asymmetric Res



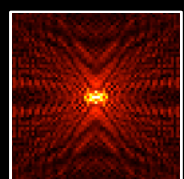
32x64



Pixels are square, but they shouldn’t be.

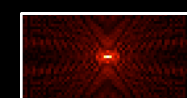
Asymmetric Resolution

Low-Res Data



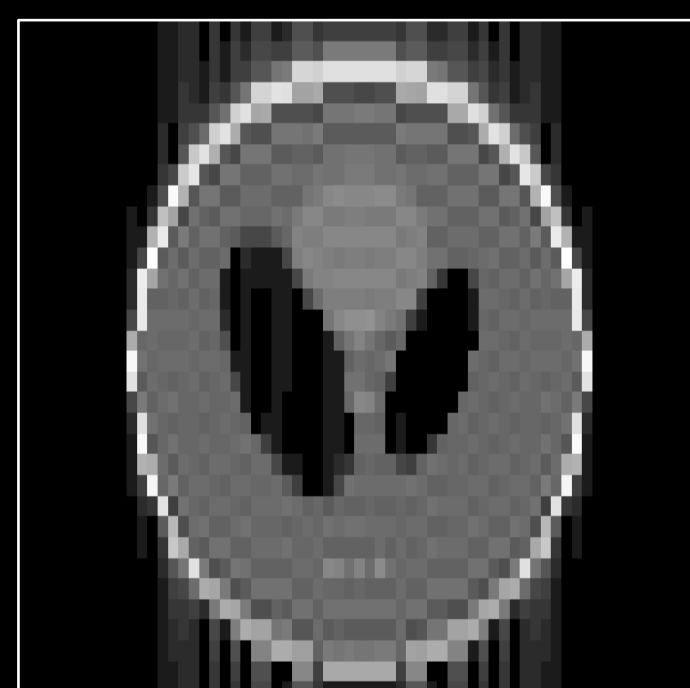
64x64

Asymmetric Res

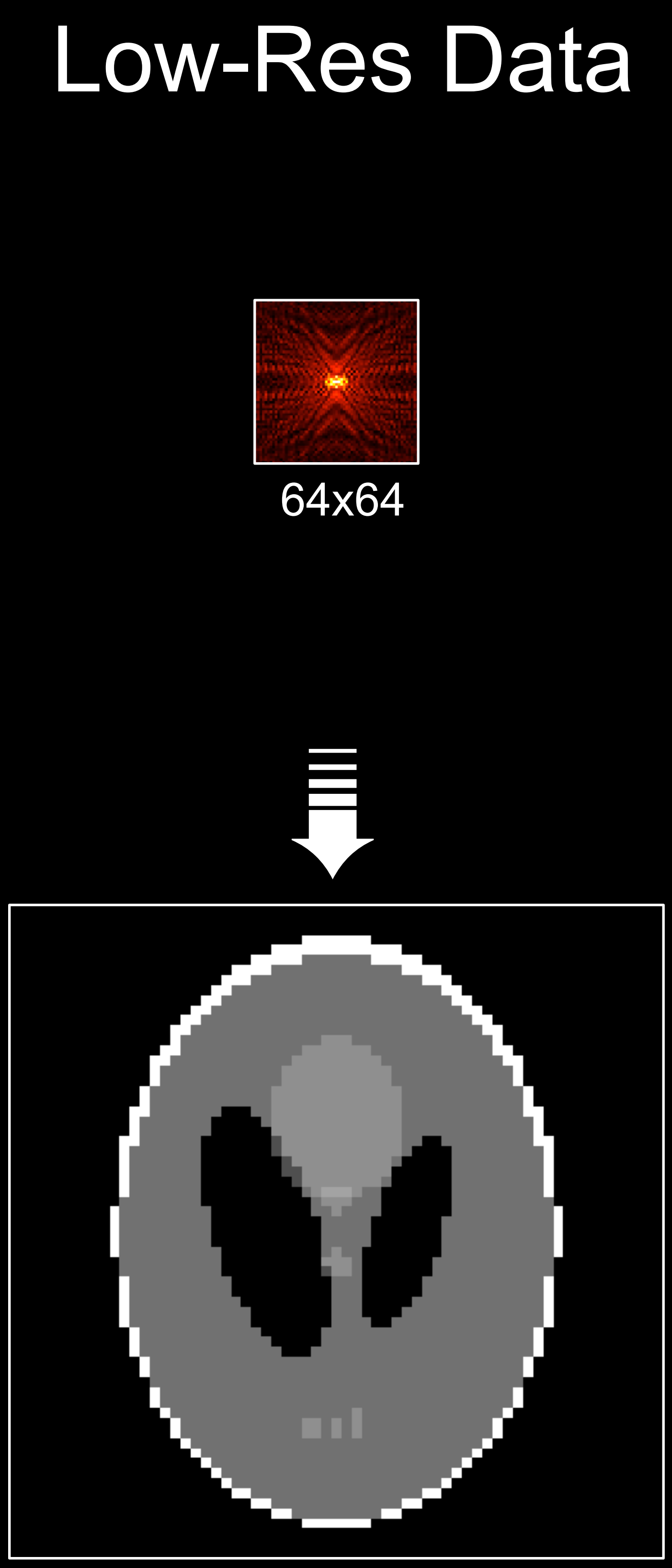


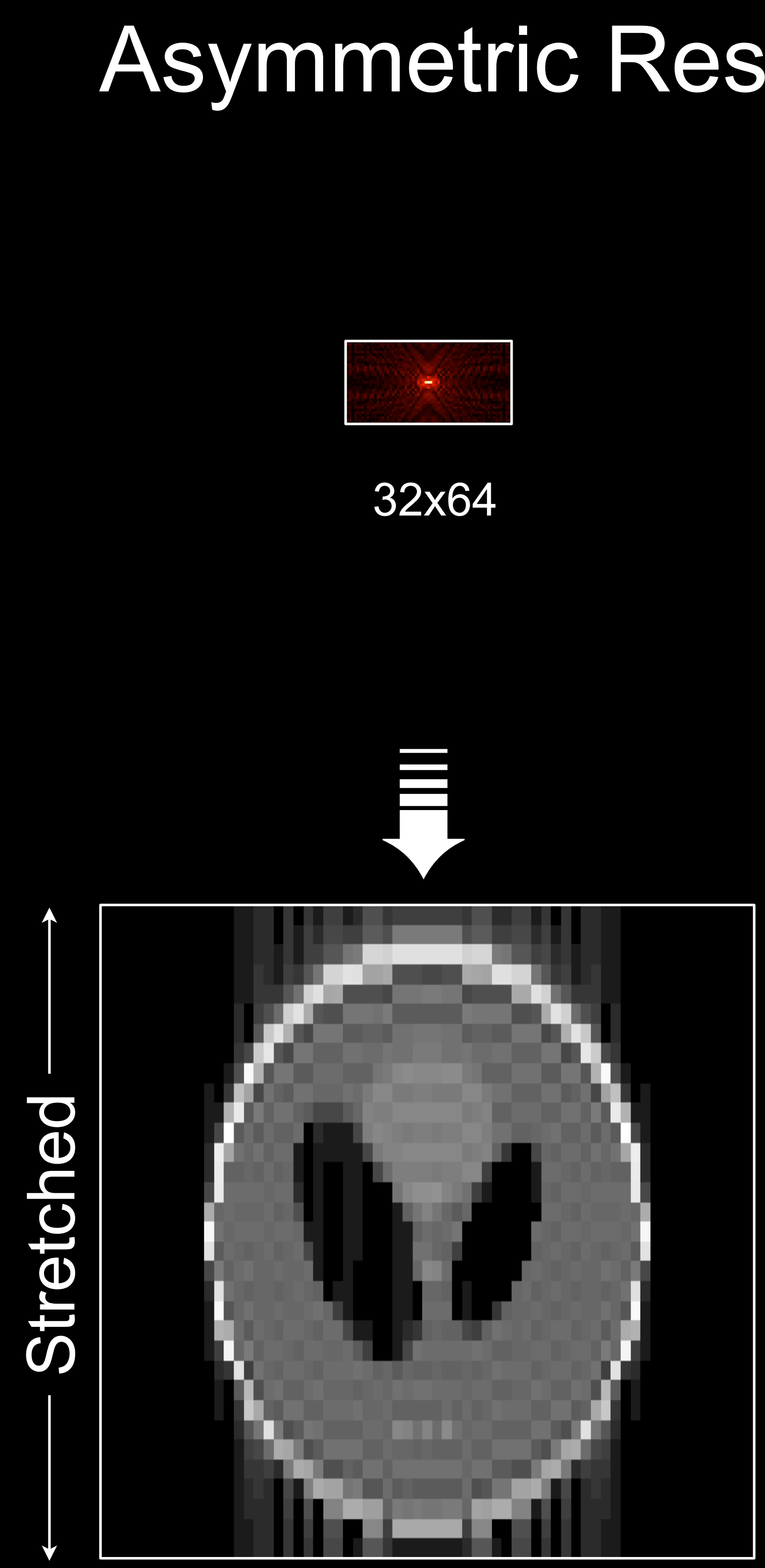
32x64

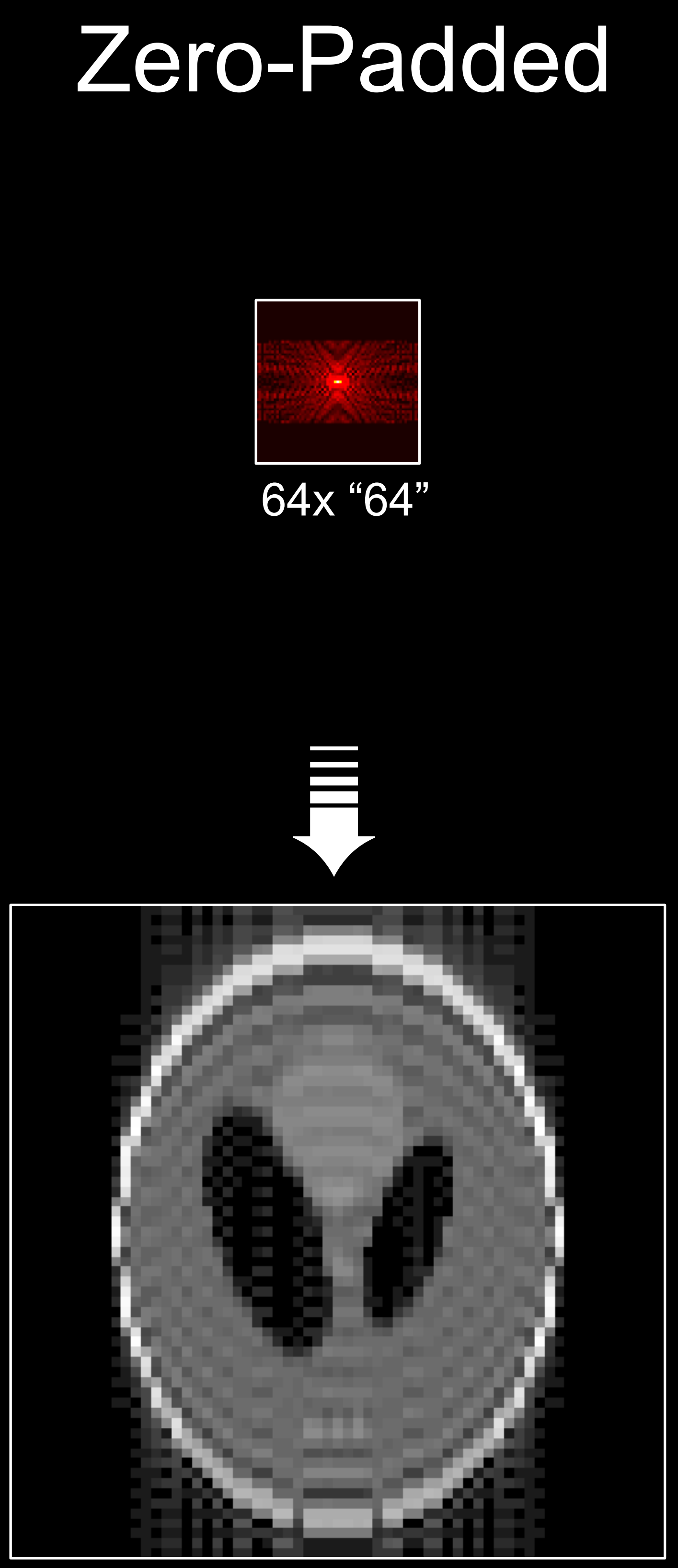
Stretched



Asymmetric Resolution





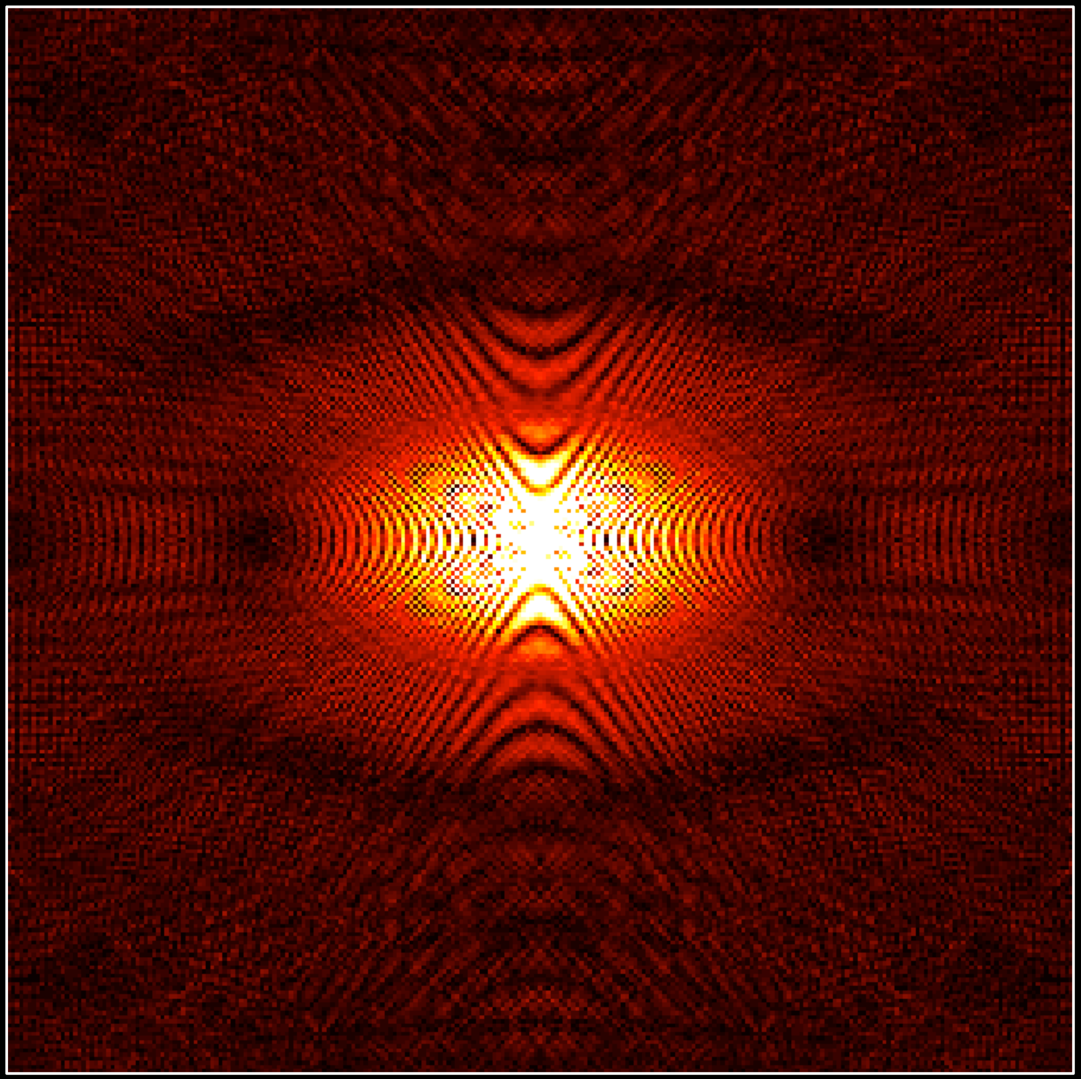


Windowed Reconstruction to Reduce Gibb's Ringing

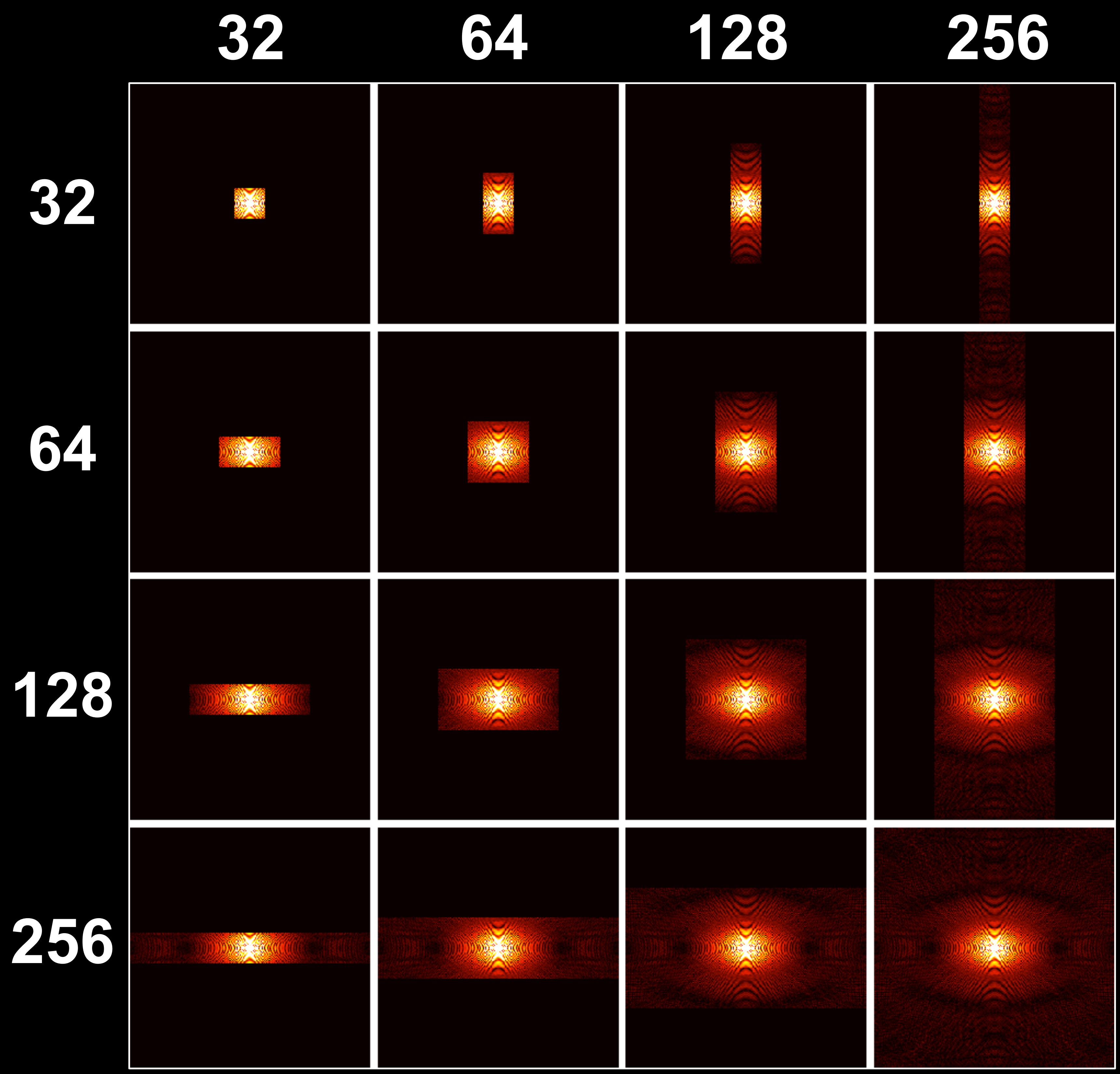
Gibb's Ringing

- Spurious ringing around sharp edges
- Max/Min overshoot is ~9% of the intensity discontinuity
  - Independent of the # of recon points
  - Frequency of ringing increases as # of recon points increases
    - Ringing becomes less apparent
- Result of truncating the Fourier series model as a consequence of finite sampling
- Can reduce by:
  - Acquiring more data
  - Filtering the data to reduce oscillations in the PSF

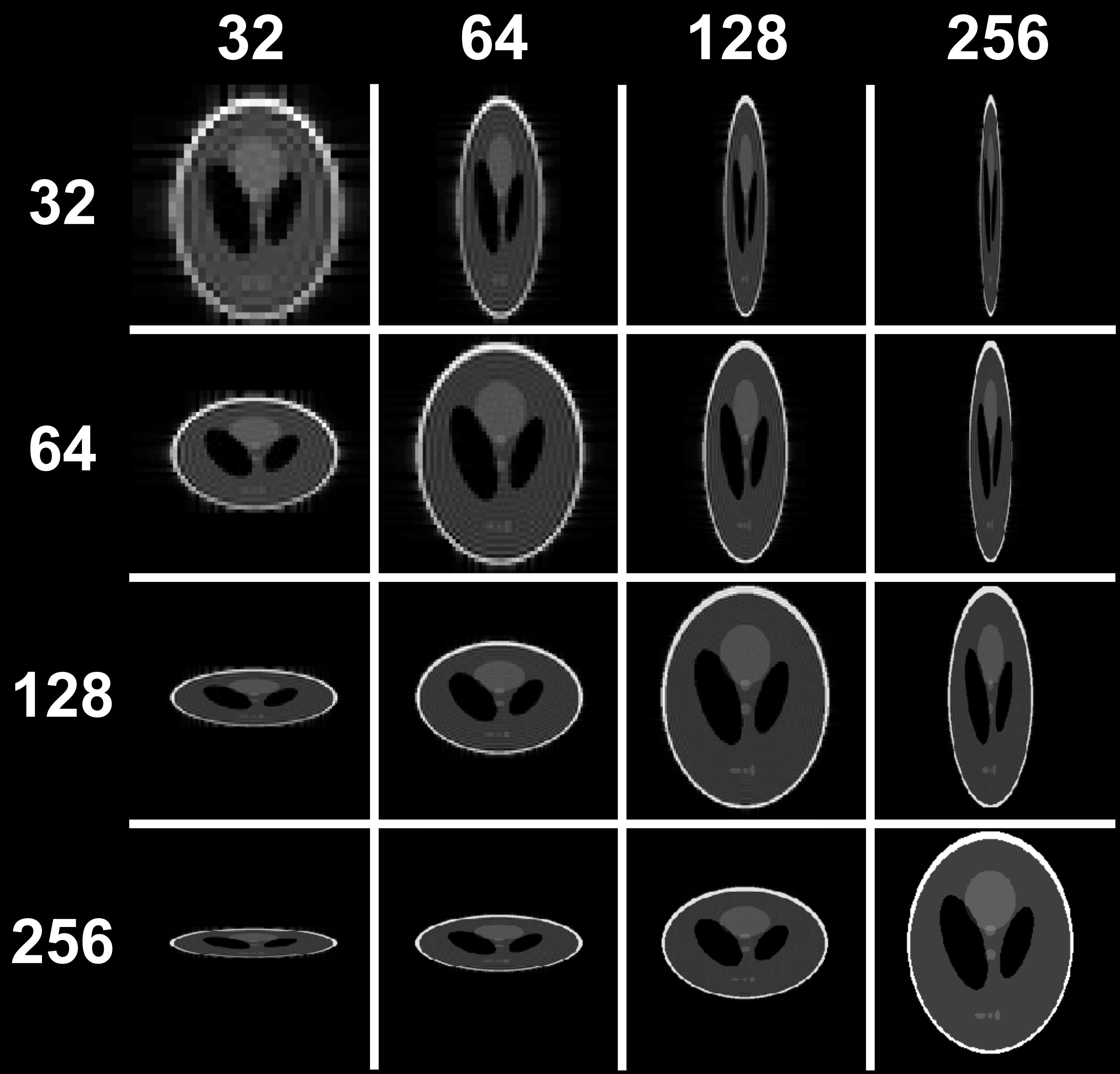
Shepp-Logan Phantom



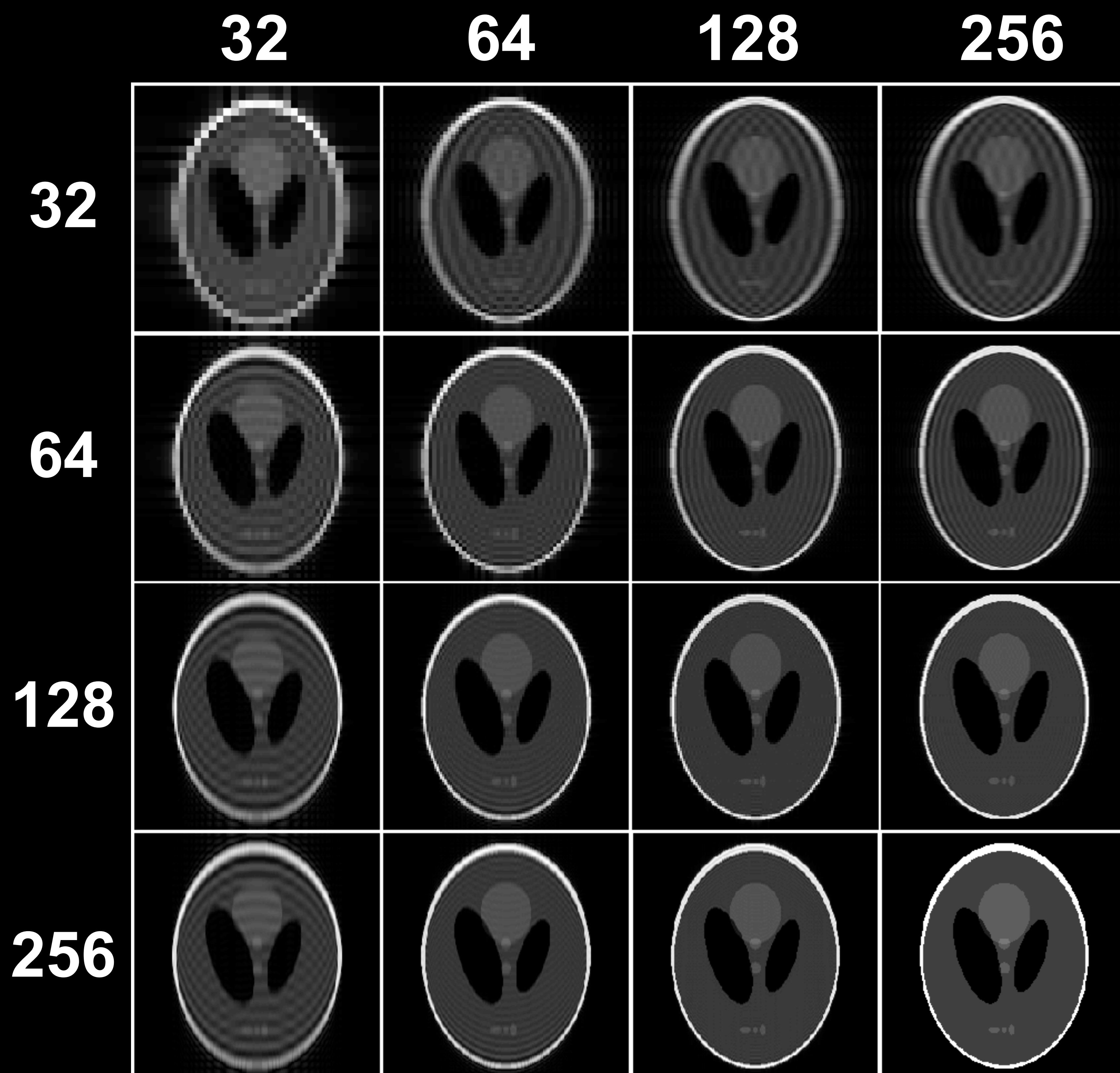
Gibb's Ringing



Gibb's Ringing



Zero-Pad



Windowed Reconstruction

$$\hat{I}(x) = \Delta k \sum_{n=-N/2}^{N/2-1} S\left(n\Delta k\right) e^{i2\pi n\Delta k x}$$

Fourier reconstruction

Windowed Reconstruction

$$\hat{I}(x) = \Delta k \sum_{n=-N/2}^{N/2-1} S\left(n\Delta k\right) e^{i2\pi n\Delta k x}$$

Fourier reconstruction

$$\hat{I}(x) = \Delta k \sum_{n=-N/2}^{N/2-1} S\left(n\Delta k\right) w_n e^{i2\pi n\Delta k x}$$ Eqn. 6.21

Windowed Fourier reconstruction

k-space filter/window function

Windowed Reconstruction

$$\hat{I}(x) = I(x) * h(x)$$

Image

Object

Point Spread Function

Windowed Reconstruction

$$\hat{I}\left(x\right) = I\left(x\right) * h\left(x\right)$$

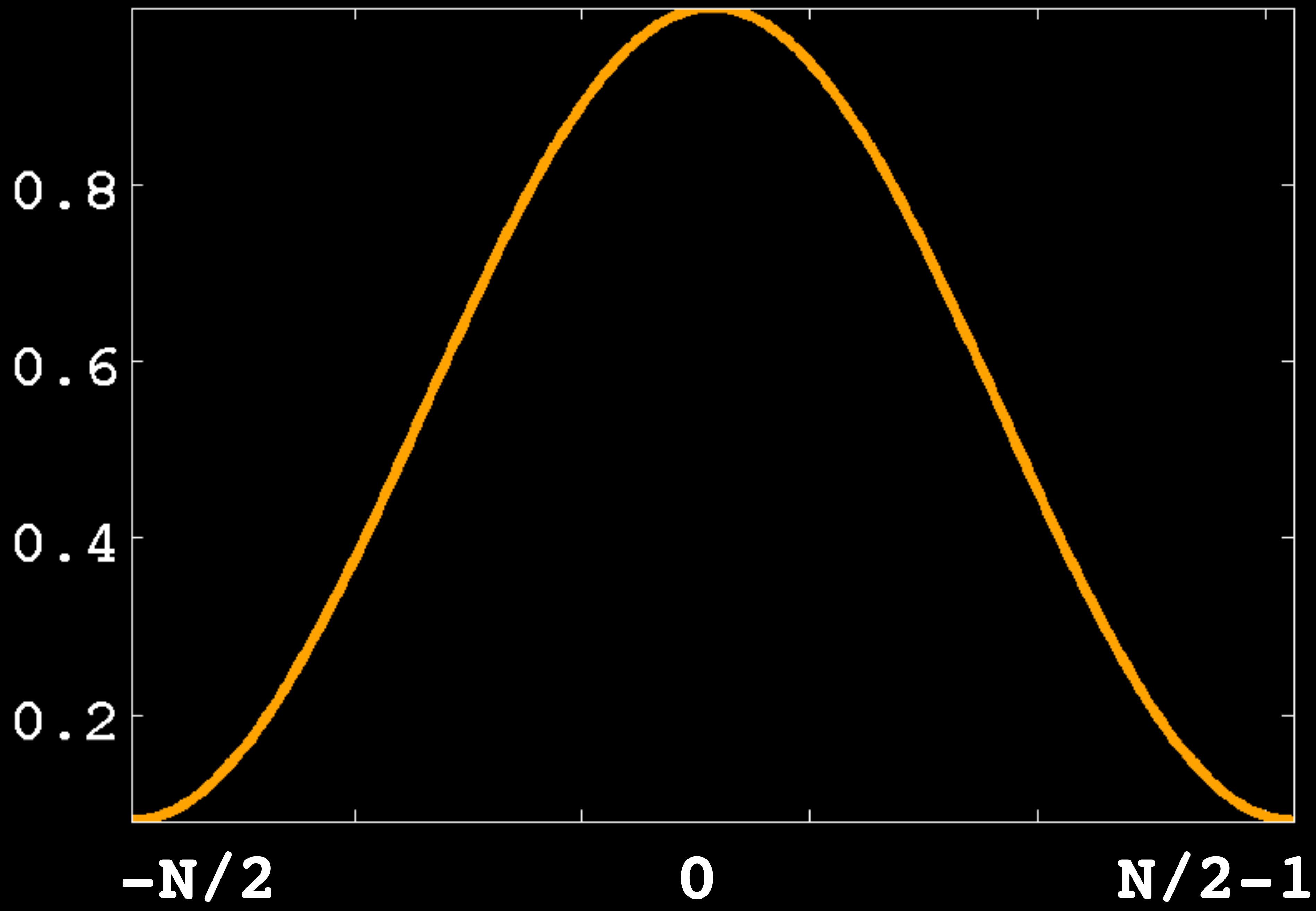
Set This To
δ-function

Point Spread Function for a windowed Fourier reconstruction.

$$h(x) = \Delta k \sum_{n=-N/2}^{N/2-1} w_n e^{i2\pi n\Delta k x}$$

Hamming Filter - 1D

$$w(n) \triangleq \begin{cases} 0.54 + 0.46\cos(2\pi\frac{n}{N}) & -N/2 \leq n \leq N/2-1 \\ 0 & \text{otherwise} \end{cases}$$



Windowed Reconstruction

FWHM PSF for a Hamming windowed Fourier reconstruction.

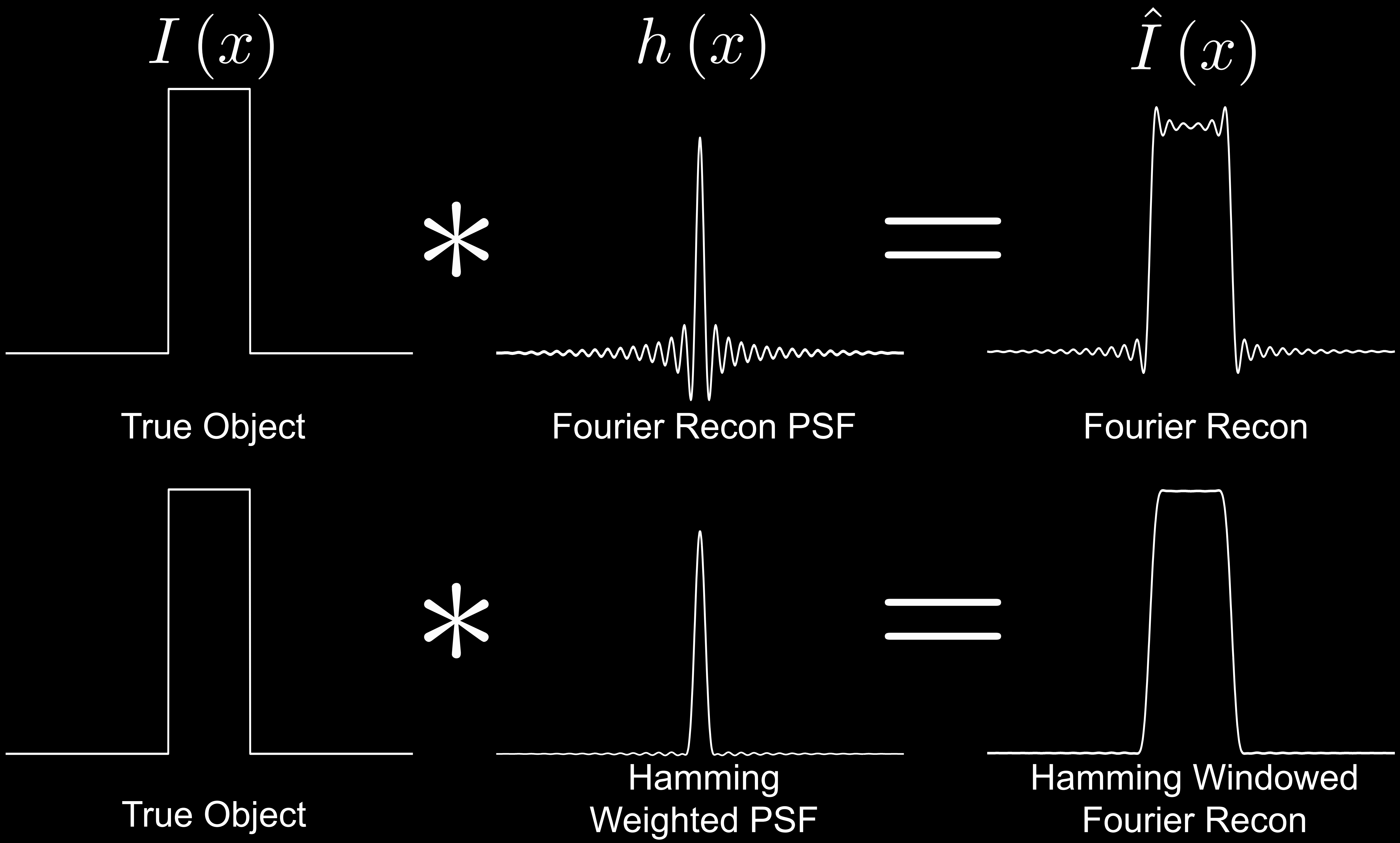
$$W_h = \left(\sum_{m=-N/2}^{N/2-1} (w_m / w_0)\, \Delta k \right)^{-1}$$

In general $w_m \leq w_0$, therefore

$$W_h \geq \frac{1}{N \Delta k}$$

Hamming windowed Fourier reconstruction suppresses ringing, but reduces effective spatial resolution.

Windowed Reconstruction



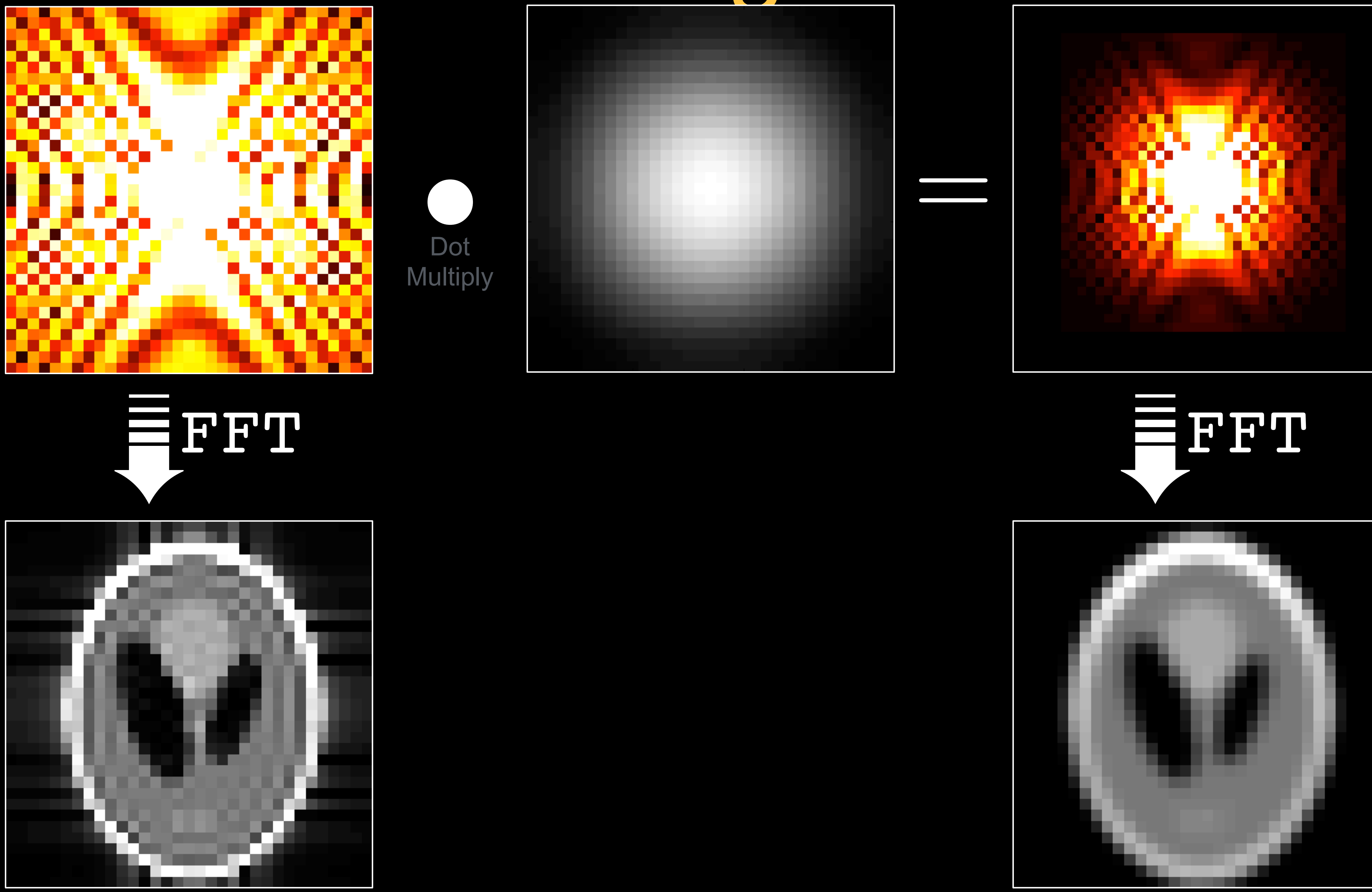
Windowed Reconstruction

- Fourier transform properties
  - Convolution in the image domain
    is equivalent to
    multiplication in the frequency domain
    (and vice versa)

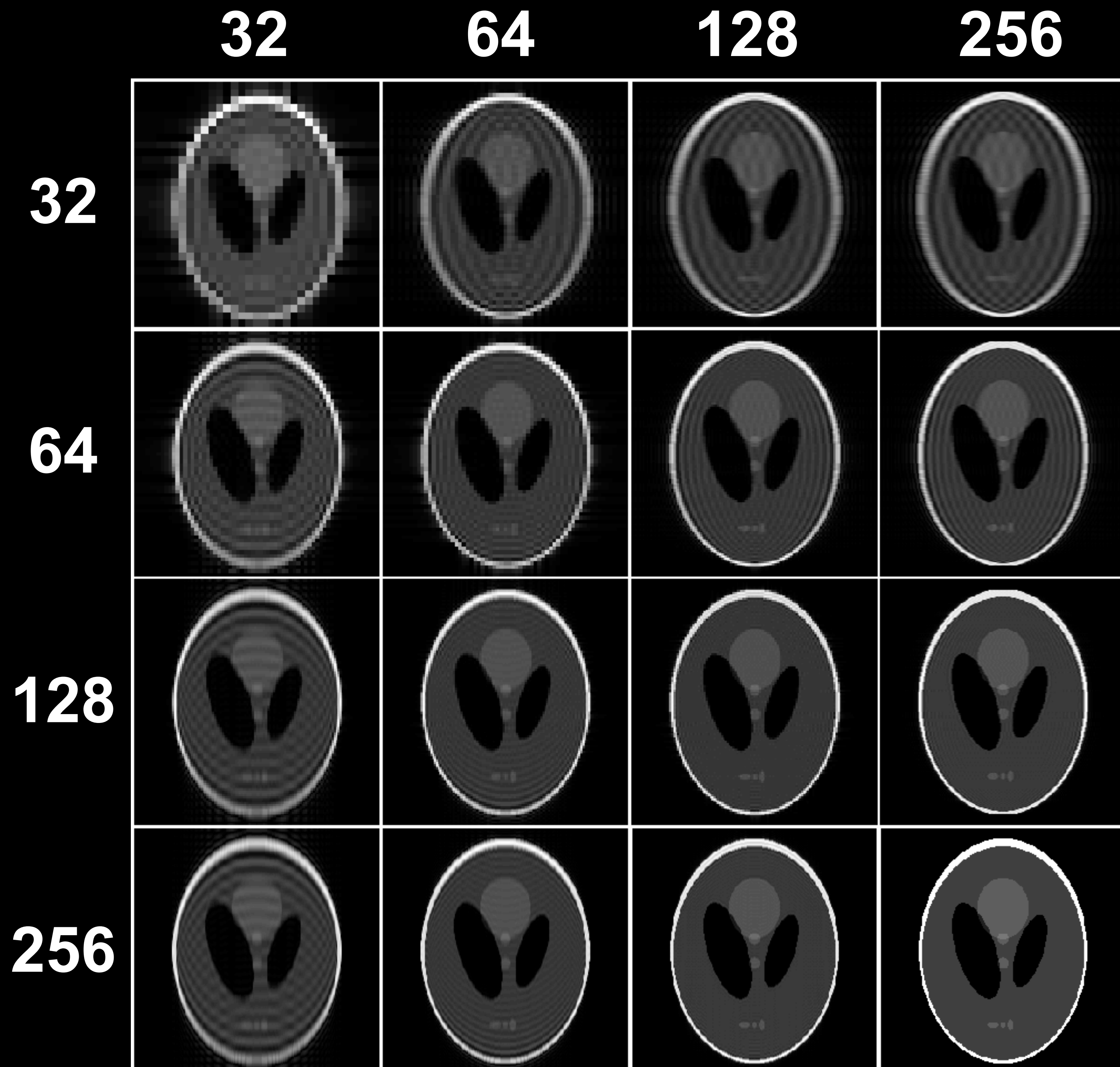
Hamming Filter - 2D

$$W(n) \triangleq w(n) \otimes w(n)$$

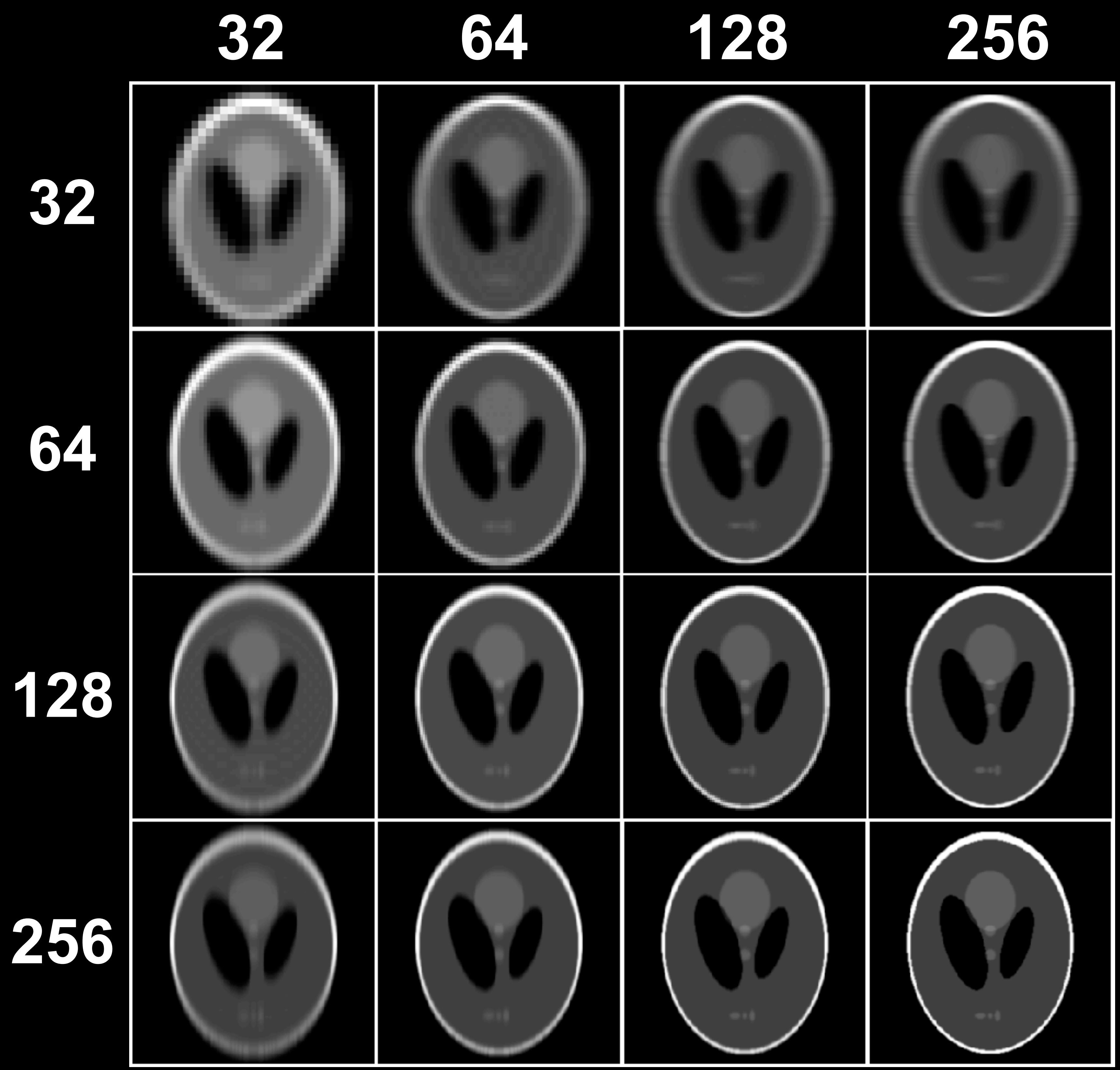
Hamming Filter



Zero-Pad

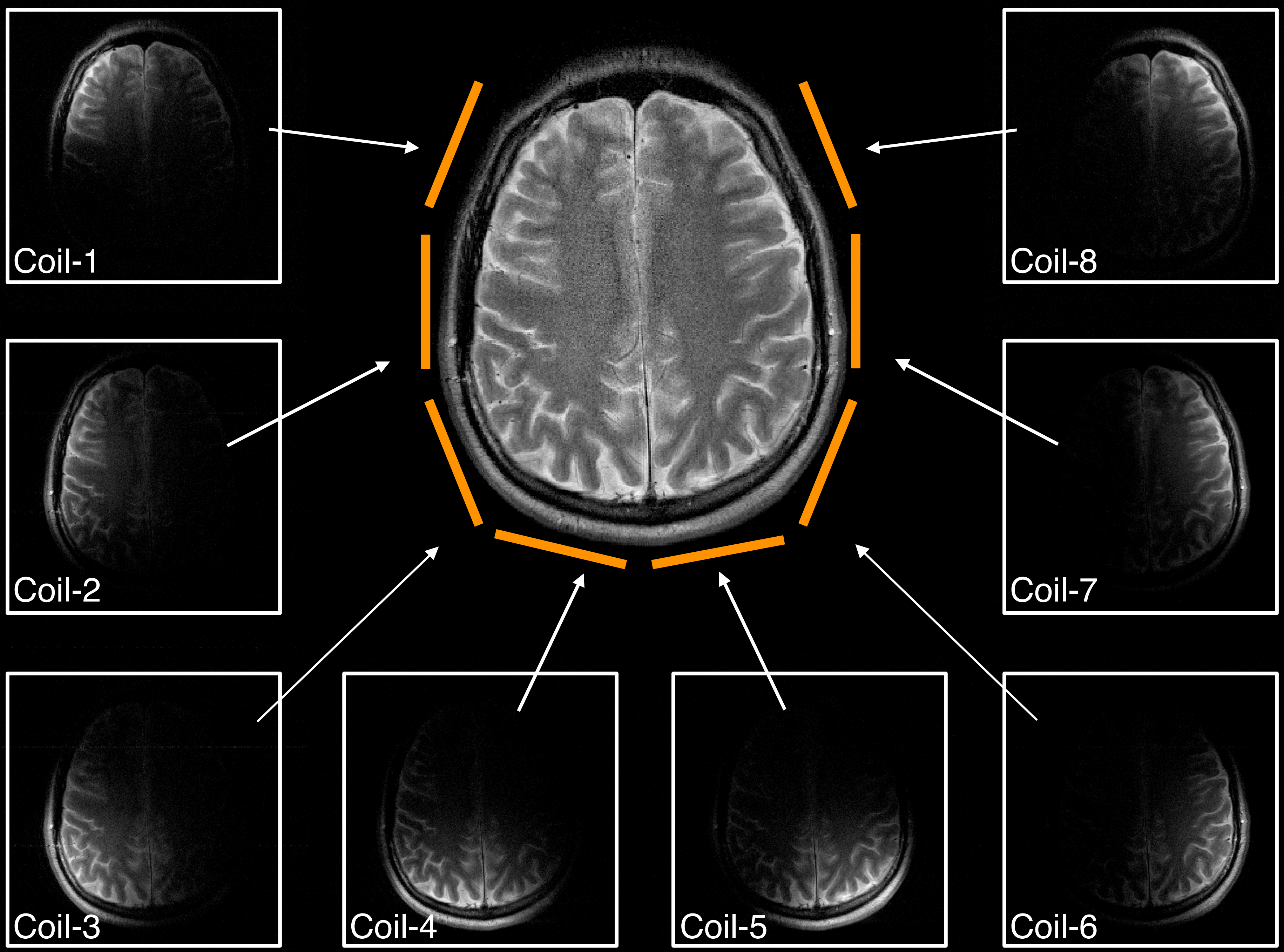


Hamming Window & Zero-Pad



Multi-Channel (Coil) Reconstruction

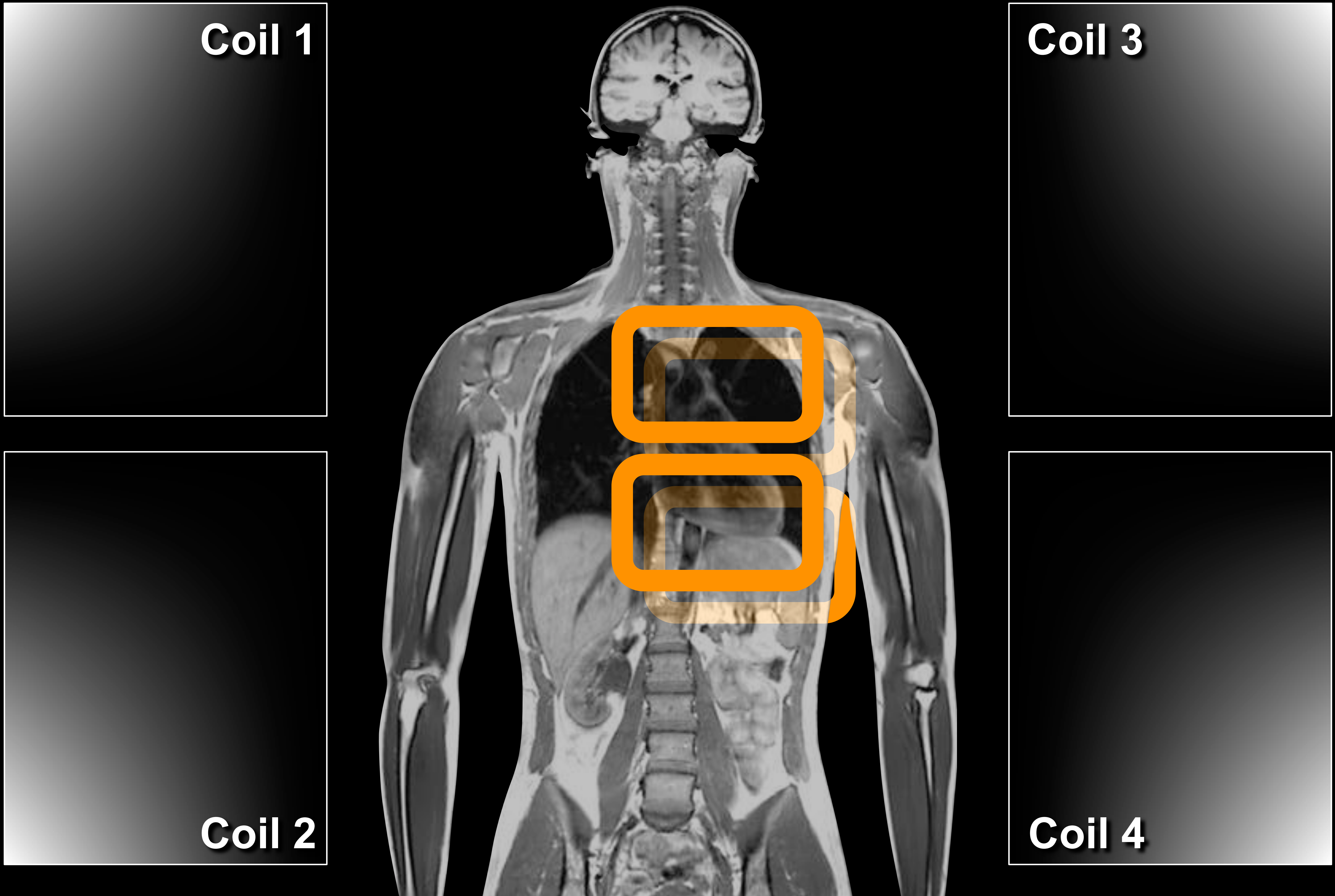
8-Channel Head Coil



Each coil element (channel) has a unique sensitivity profile – $\vec{B}_r\,(\vec{r})$

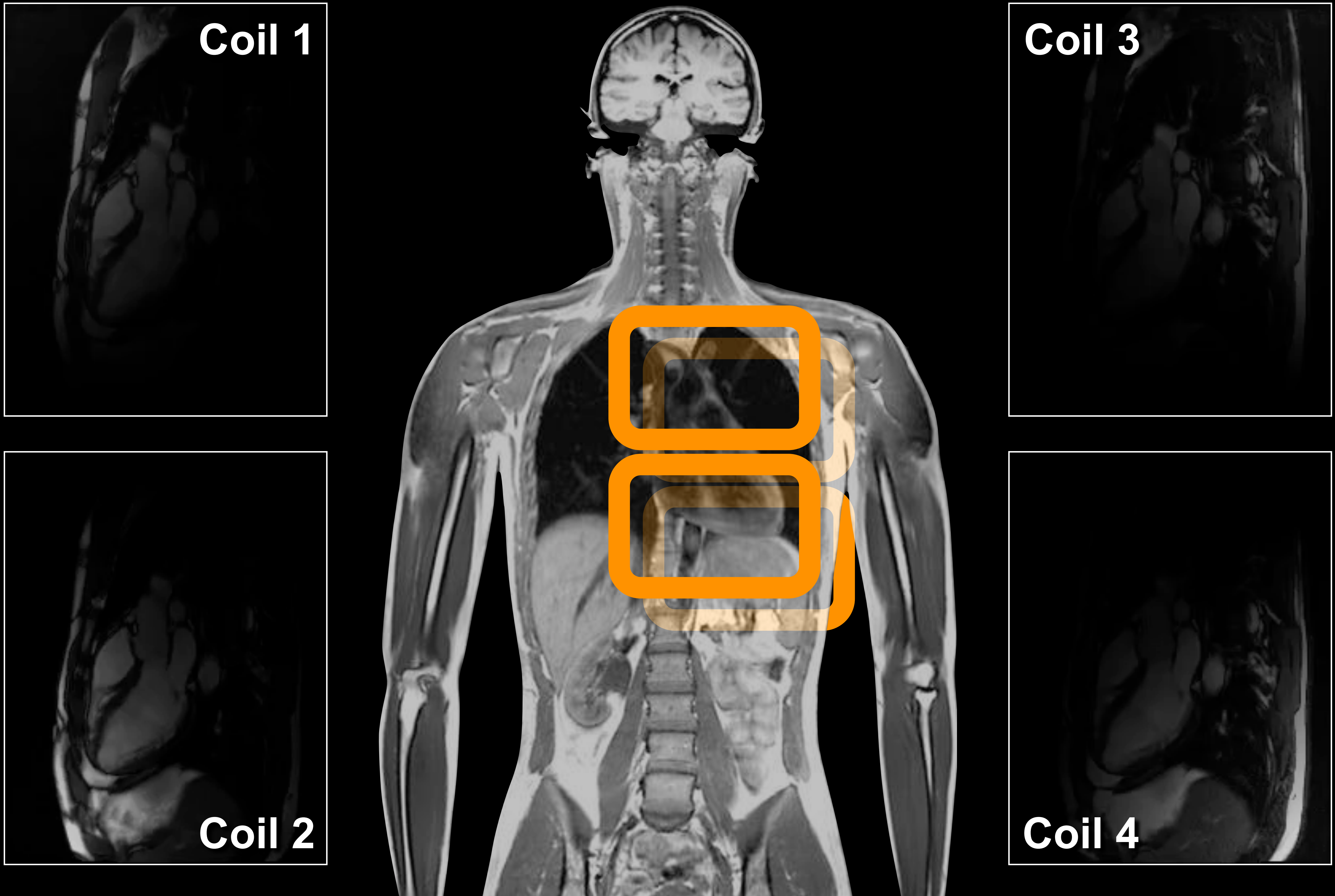
4-Channel Cardiac Coil

Each coil element (channel) has a unique sensitivity profile – $\vec{B}_r\,(\vec{r})$

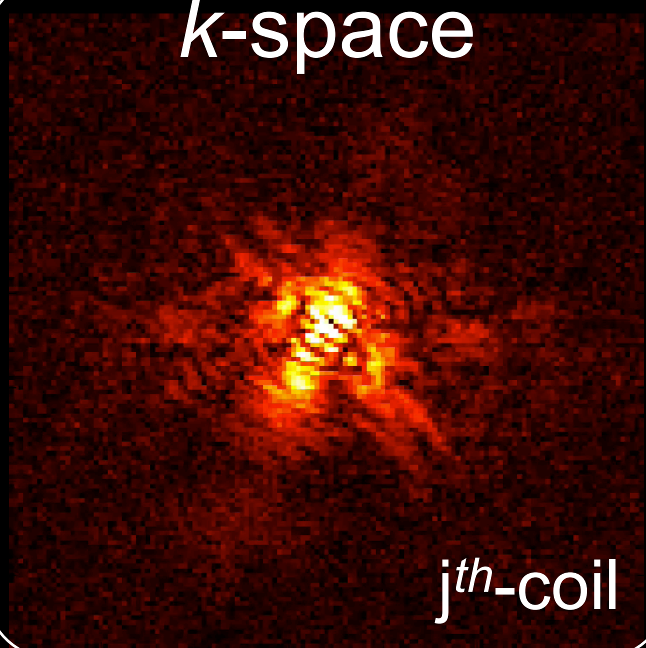


4-Channel Cardiac Coil

Each coil element (channel) has a unique sensitivity profile – $\vec{B}_r(\vec{r})$



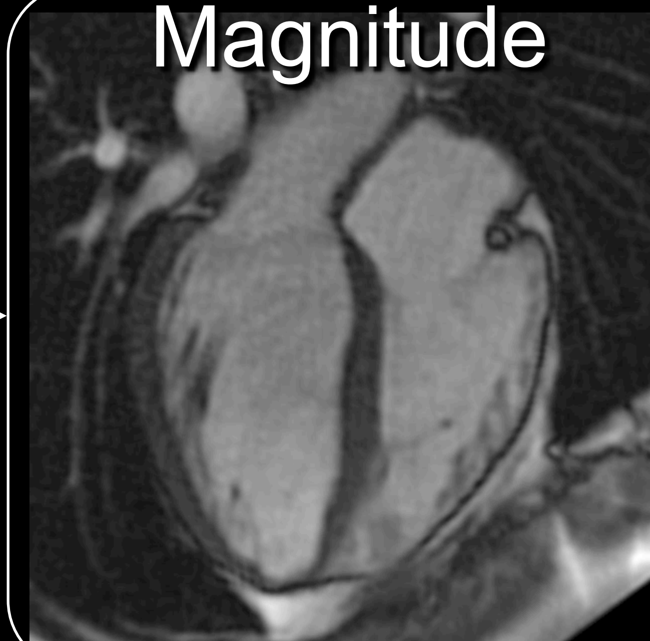
Multi-Coil Reconstruction



FFT-1

RMS

$$I(\vec{r}) = \sqrt{\sum_j \left(\frac{|I_j(\vec{r})|^2}{\sigma_j^2} \right)}$$



$I(\vec{r}) \rightarrow$ Final ***magnitude*** image

$I_j(\vec{r}) \rightarrow$ Image from j*th* coil

$\sigma_j^2 \rightarrow$ Noise variance

- Depends on coil loading
- Proximity to patient
- Measured with “noise scan”
- Weights each coil’s contribution

Thanks!

- Next: fast imaging, advanced recon
- Acknowledgments
  - Dr. Daniel Ennis
  - Dr. Peng Hu
  - Dr. Kyung Sung

Holden H. Wu, Ph.D.

HoldenWu@mednet.ucla.edu

http://mrrl.ucla.edu/wulab