

Image Reconstruction

Partial k-space Reconstruction

M229 Advanced Topics in MRI

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Class Business

- Homework 1 grading is done
- Project proposal comments will be out by this Friday
- Office hours
 - Instructors: Wed 3-5pm
email beforehand would be helpful

Today's Topics

- Fourier transform symmetries
 - Odd and Even functions
- Motivation for partial k-space recon
- Partial k-space recon methods
 - Direct method (Homodyne)
 - Iterative method (POCS)
- MATLAB code demo

Even and Odd Functions

- function f is even (or symmetric) when
$$f(x) = f(-x)$$
- function f is odd (or antisymmetric) when
$$f(x) = -f(-x)$$

Even and Odd Functions

- Any function can be written as a sum of even and odd functions

$$\begin{aligned} f(x) &= \frac{1}{2}[f(x) + f(-x) - f(-x) + f(x)] \\ &= \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)] \\ &\quad \underbrace{\hspace{10em}}_{f_e(x)} \quad \underbrace{\hspace{10em}}_{f_o(x)} \end{aligned}$$

Even and Odd Functions

- The integral of the product of odd and even functions is zero

$$\begin{aligned} &\int_{-\infty}^{\infty} f_e(x) f_o(x) dx \\ &= \int_{-\infty}^0 f_e(x) f_o(x) dx + \int_0^{\infty} f_e(x) f_o(x) dx \\ &= \int_0^{\infty} [f_e(-x) f_o(-x) dx + f_e(x) f_o(x)] dx \end{aligned}$$

Fourier Transform Symmetry

$$F(f) = \int_{-\infty}^{\infty} f(x)e^{-j2\pi xf} dx$$

$$F(f) = \int_{-\infty}^{\infty} f(x) \cos(2\pi xf) dx - j \int_{-\infty}^{\infty} f(x) \sin(2\pi xf) dx$$

$$F(f) = \int_{-\infty}^{\infty} f_e(x) \cos(2\pi xf) dx + \int_{-\infty}^{\infty} f_o(x) \cos(2\pi xf) dx$$
$$- j \int_{-\infty}^{\infty} f_e(x) \sin(2\pi xf) dx - j \int_{-\infty}^{\infty} f_o(x) \sin(2\pi xf) dx$$

Fourier Transform Symmetry

$$F(f) = \int_{-\infty}^{\infty} f_e(x) \cos(2\pi xf) dx - j \int_{-\infty}^{\infty} f_o(x) \sin(2\pi xf) dx$$

$$F(f) = F_e(f) + F_o(f)$$

real & even function?

real & odd function?

even function?

odd function?

Fourier Transform Symmetry

- Fourier transform of even part (of a real function) is real

$$FT\{f_e(x)\} = Re\{F_e(f)\}$$

- Fourier transform of even part is even

$$FT\{f_e(x)\} = F_e(f) = F_e(-f)$$

Fourier Transform Symmetry

- Fourier transform of odd part (of a real function) is imaginary

$$FT\{f_o(x)\} = Im\{F_o(f)\}$$

- Fourier transform of odd part is odd

$$FT\{f_o(x)\} = F_o(f) = -F_o(-f)$$

Hermitian Symmetry

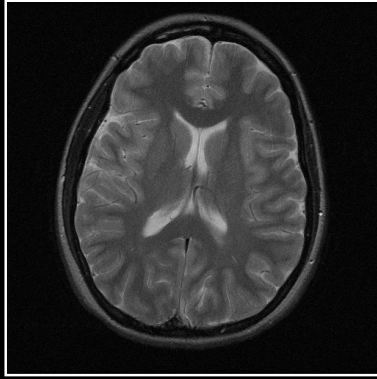
- We can summarize all four symmetries possessed by Fourier transform of a real function

To the board ...

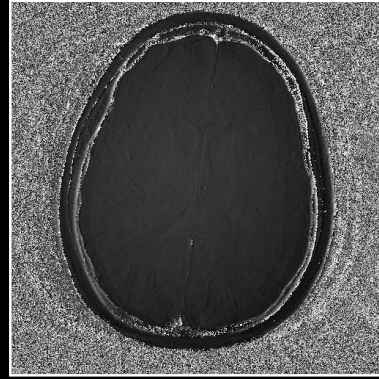
Motivation

- MR images depict the spin density as a function of position
 - If this is true, only half of k-space data will need to be collected
 - Uncollected data could be synthesized by conjugate symmetry
- However, MR images are not real-valued!
 - Partial k-space reconstruction requires some type of phase correction

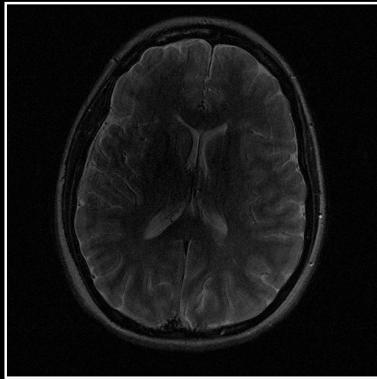
Magnitude



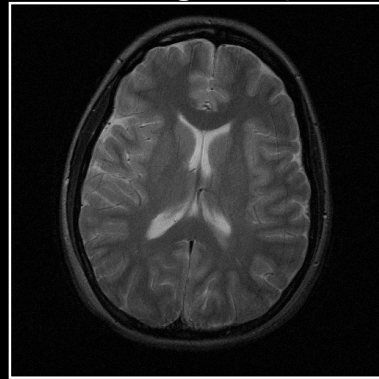
Phase



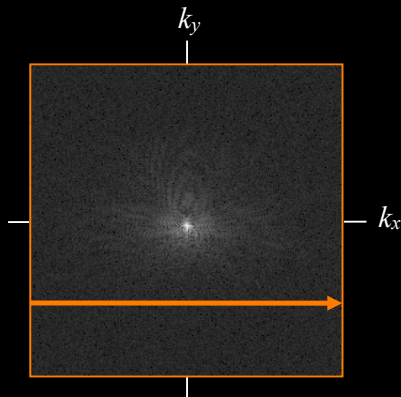
Real



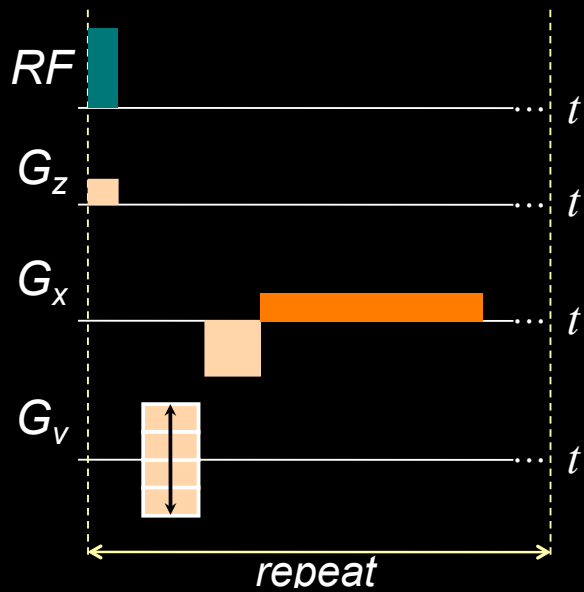
Imaginary



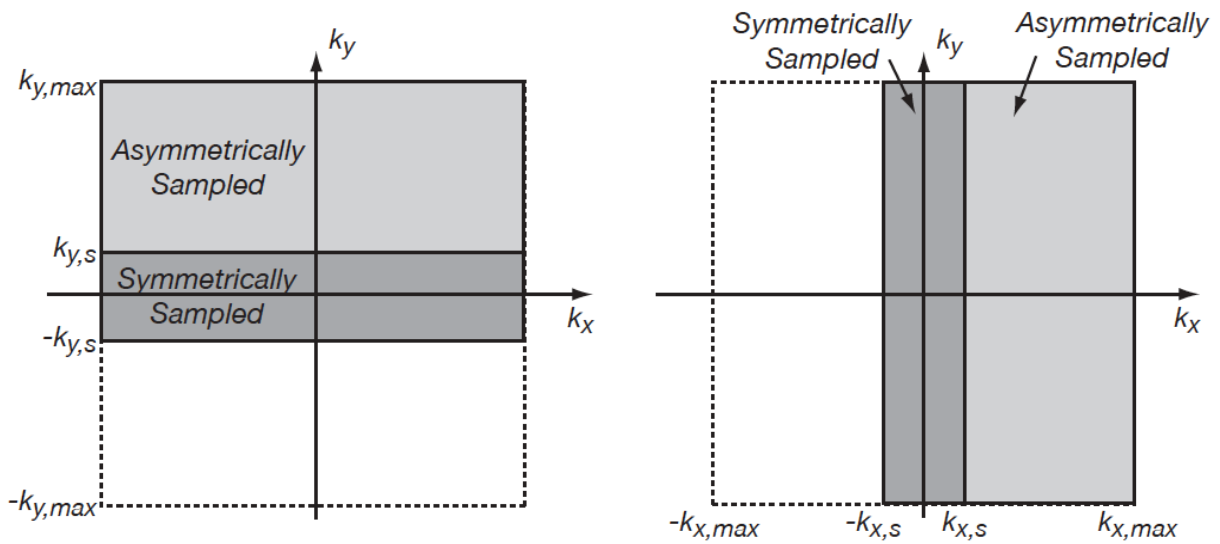
Cartesian 2D Imaging



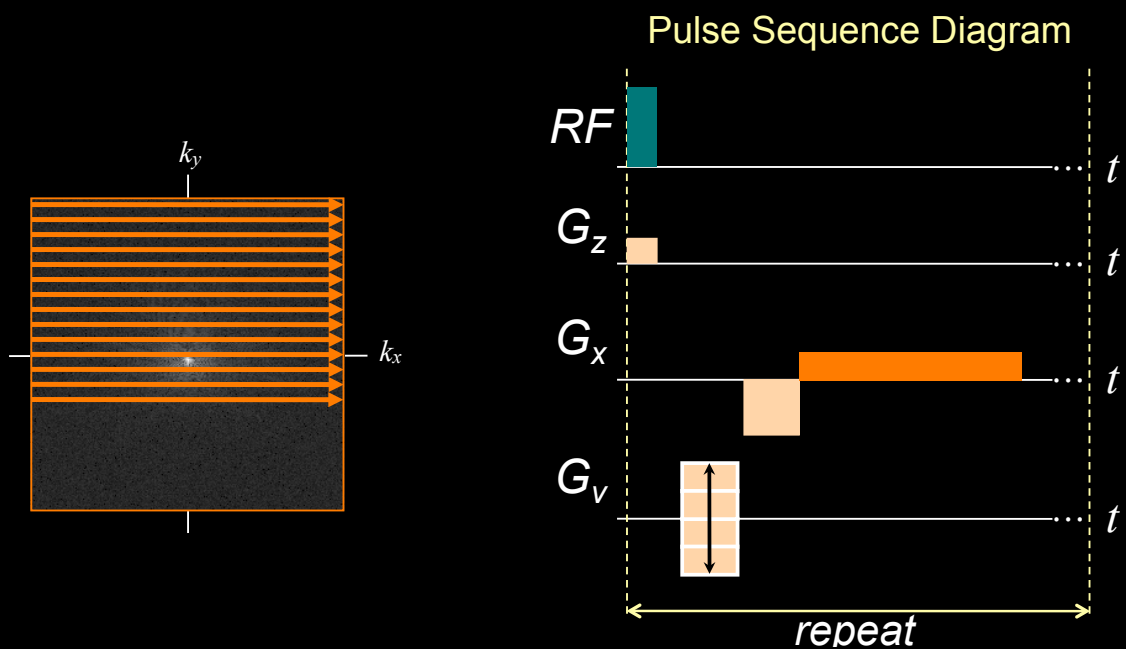
Pulse Sequence Diagram



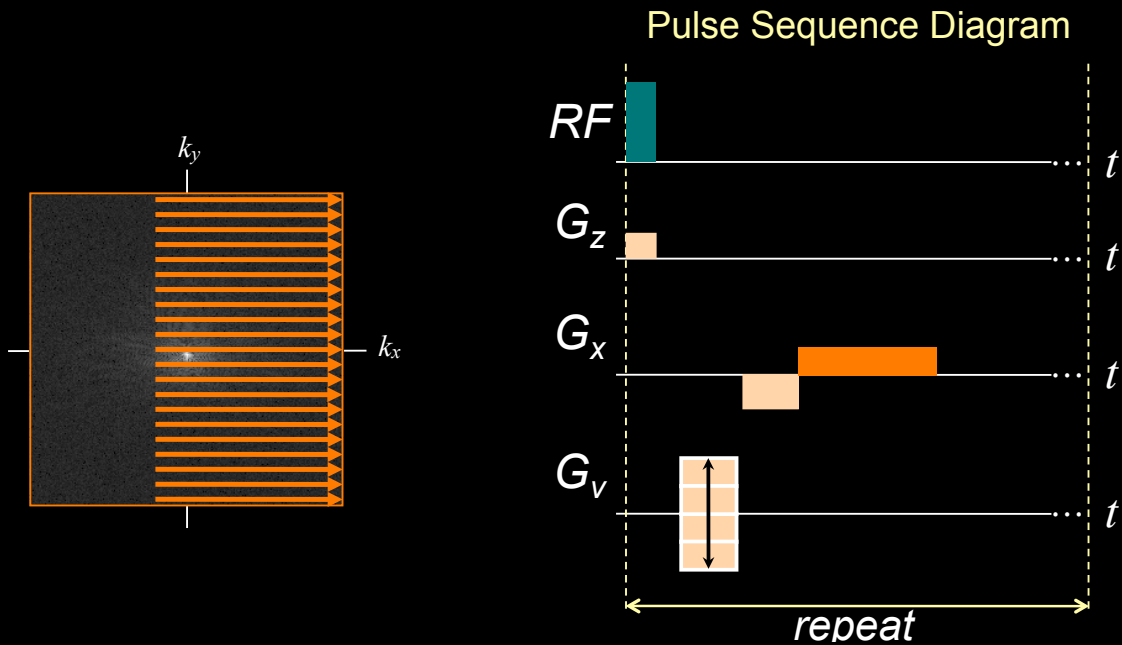
Cartesian Sampling Application



Cartesian 2D Imaging



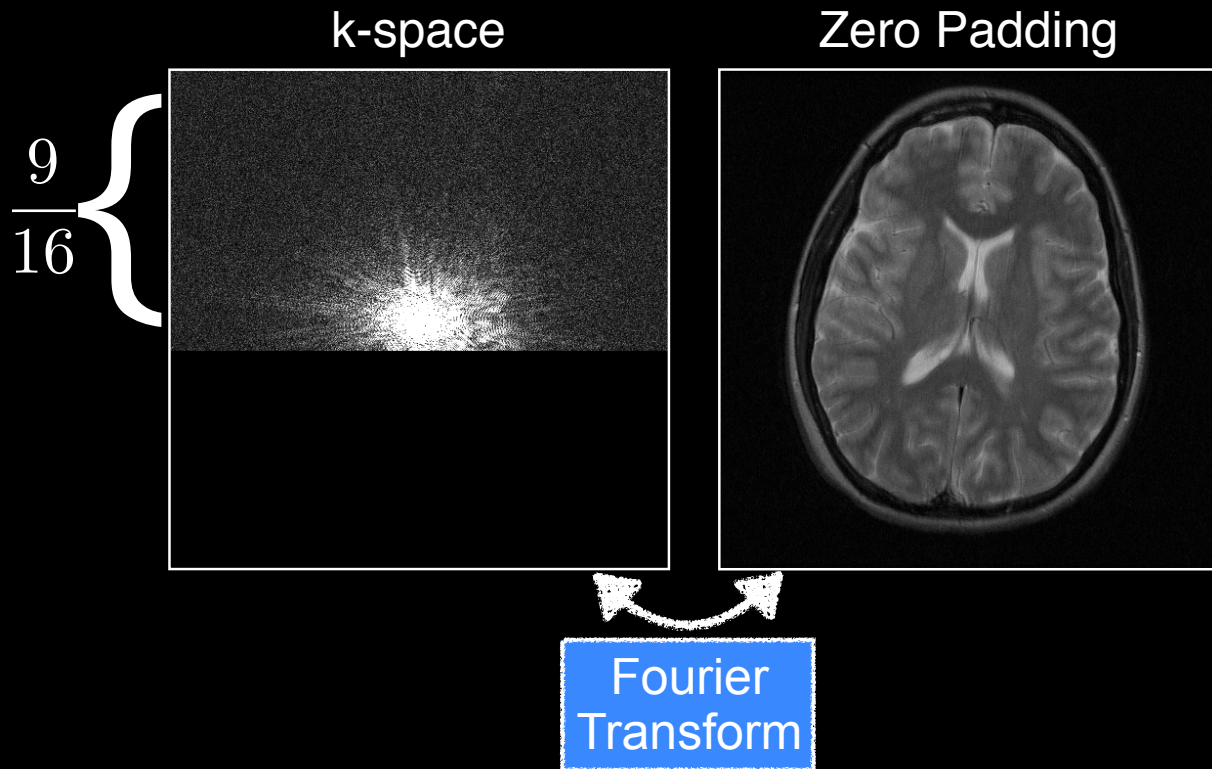
Cartesian 2D Imaging



Direct Reconstruction

- Zero padding
- Phase correction and conjugate synthesis
- Homodyne reconstruction

Trivial Recon by Zero-Padding



Zero Padding

Original

Zero Padding

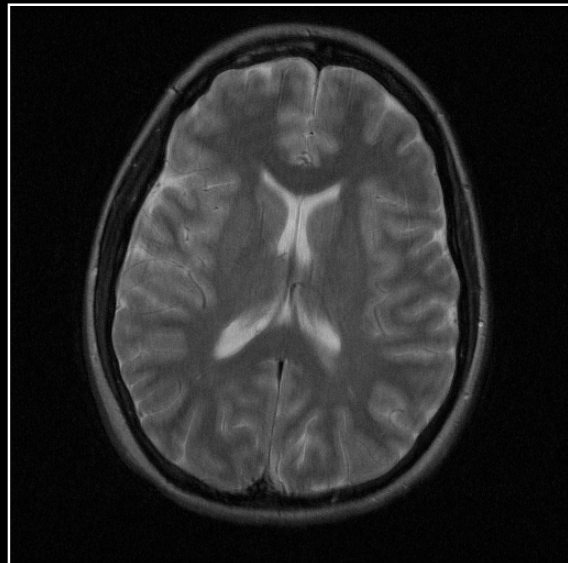
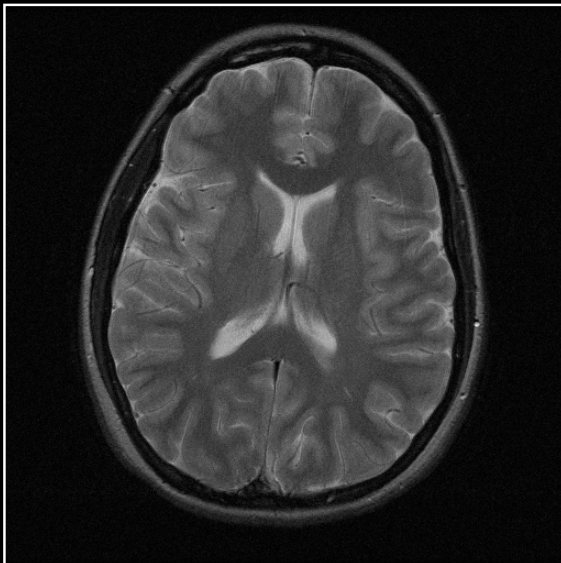
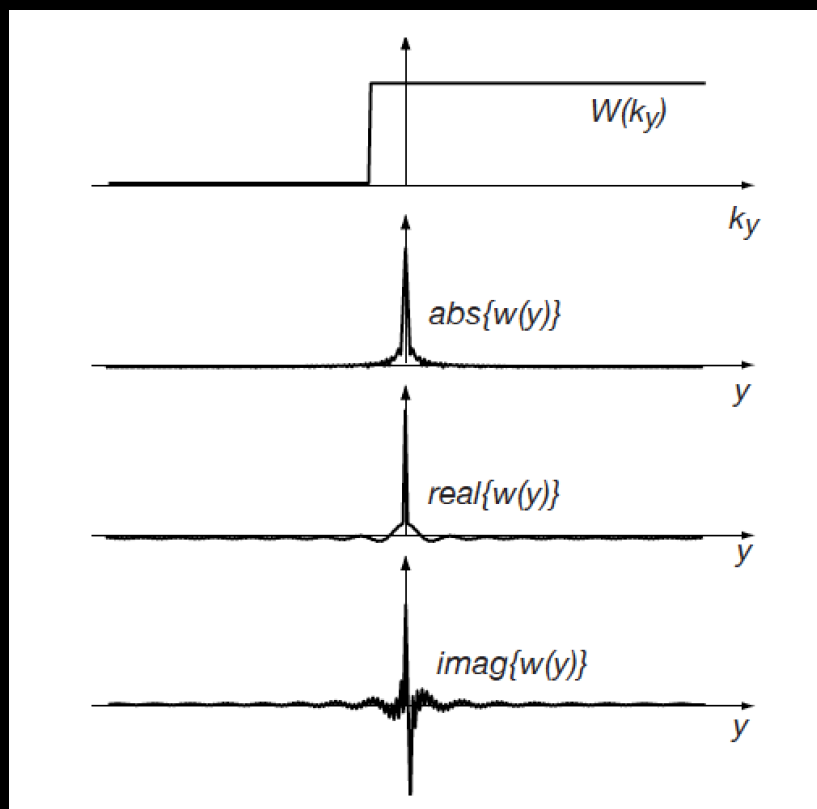


Image Artifacts by Zero Padding

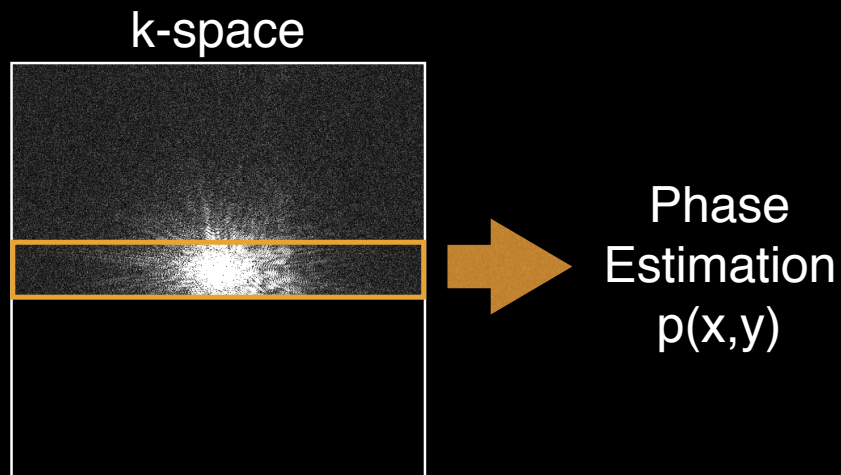
- Blurring can be identified by the product of a full k-space data set multiplied by a weighting function
- The inverse Fourier transform of this weighting function is the impulse response that produces the blurring

Image Artifacts by Zero Padding



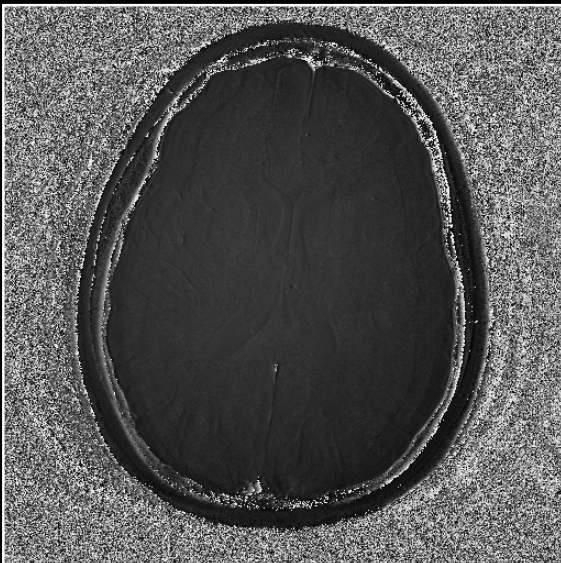
Phase Correction and Conjugate Synthesis

- Phase correction must be applied
 - Use the narrow strip of data for which we have symmetric coverage

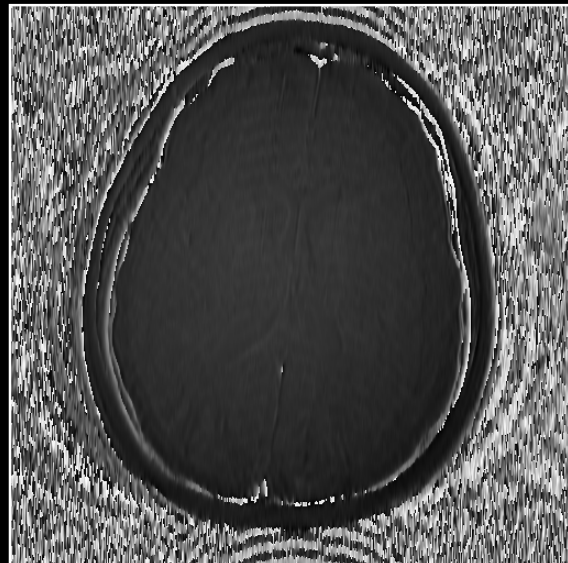


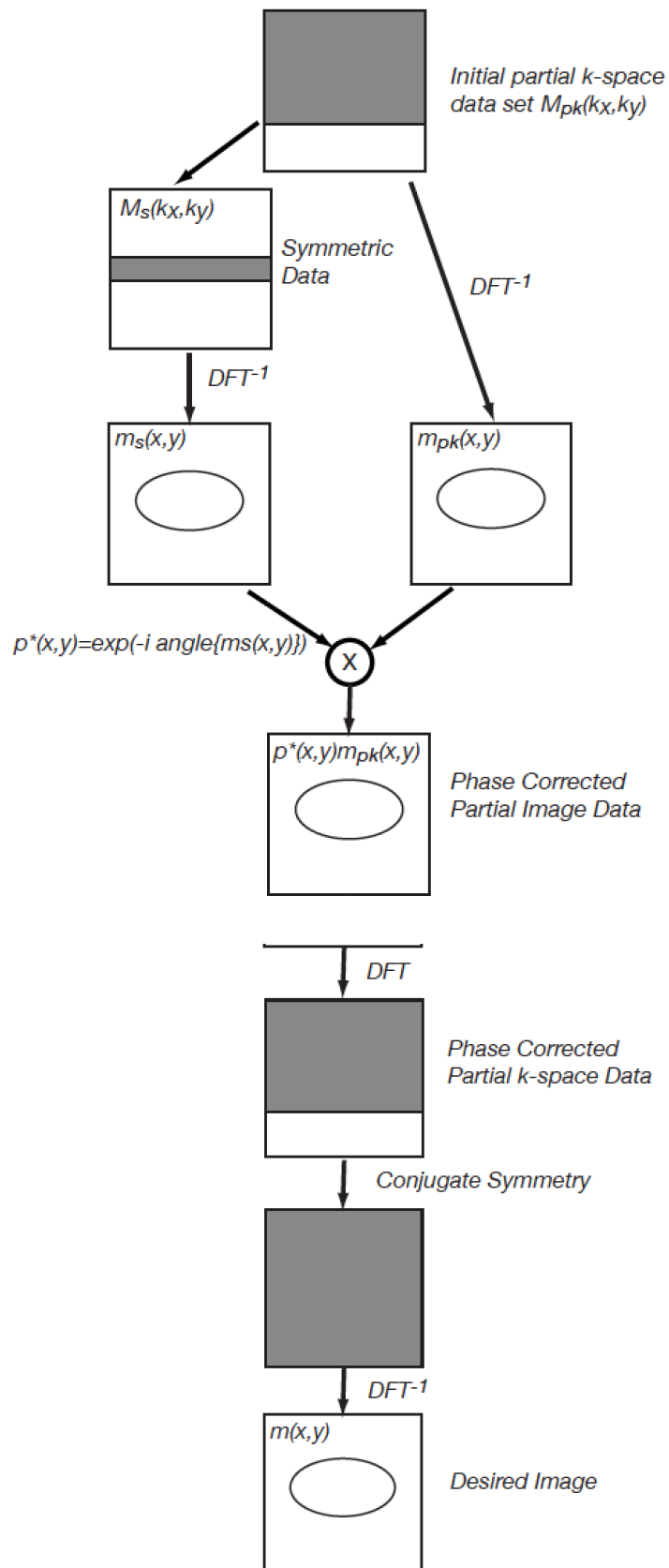
Phase Estimation

Original Phase



Estimated Phase





MATLAB Code

```
hnover = 224; % 7/16 sets to be zeros

data_pk = data;
data_pk(1+nx-hnover:end,:) = 0;

im_zp = fftshift(ifftn(fftshift(data_pk)));

data_center = data_pk;
data_center(1:hnover,:) = 0;

im_ph = fftshift(ifftn(fftshift(data_center)));

im_pc = im_zp.*exp(-1i*angle(im_ph));

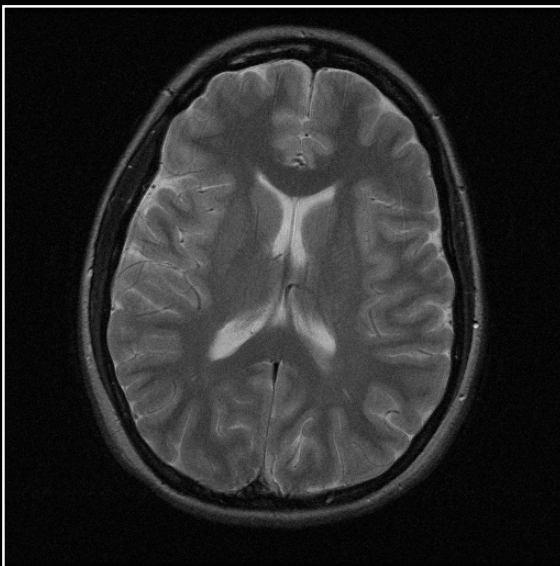
data_pc = fftshift(fftn(fftshift(im_pc)));
data_pc(1+nx-hnover:end,:) = 0;

data_pc(1+nx-hnover:end,:) = rot90(data_pc(1:hnover,:),2);

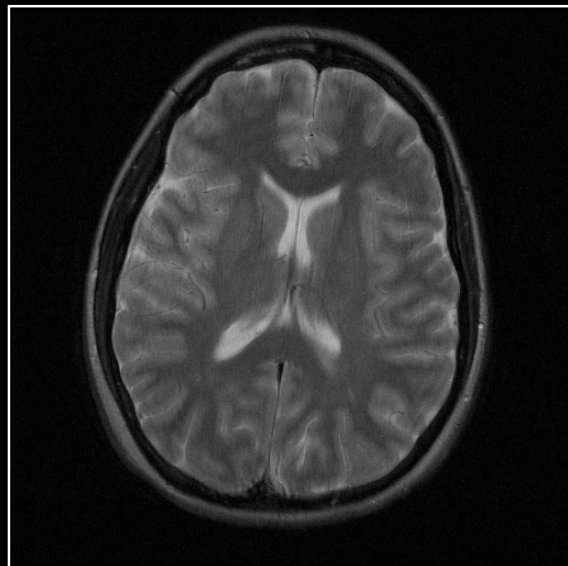
im_pc = fftshift(ifftn(fftshift(data_pc)));
```

Phase Correction and Conjugate Synthesis

Original

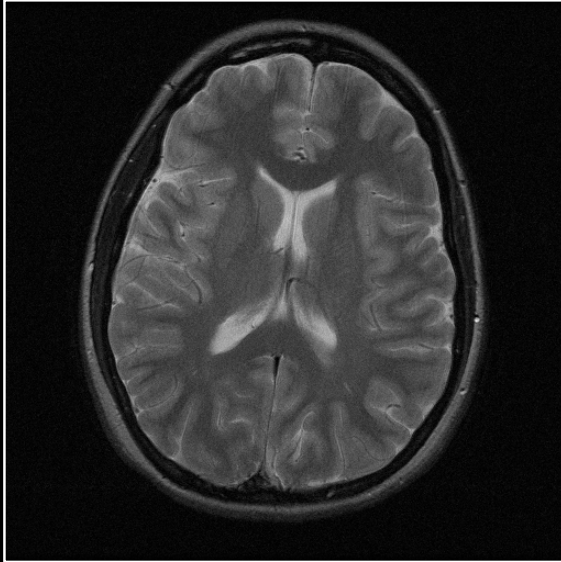


Zero Padding

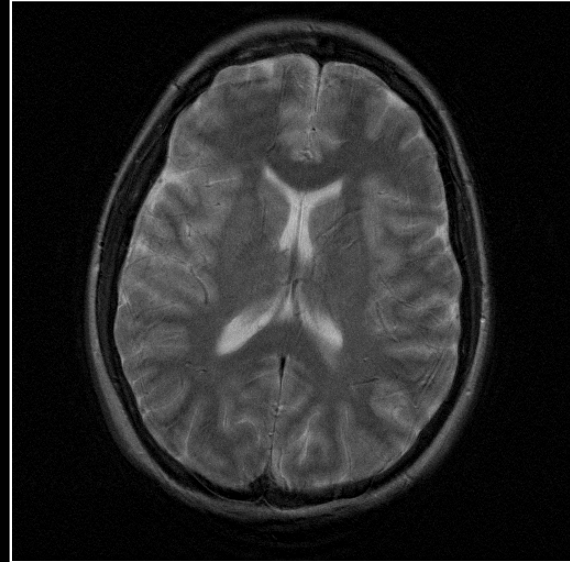


Phase Correction and Conjugate Synthesis

Original

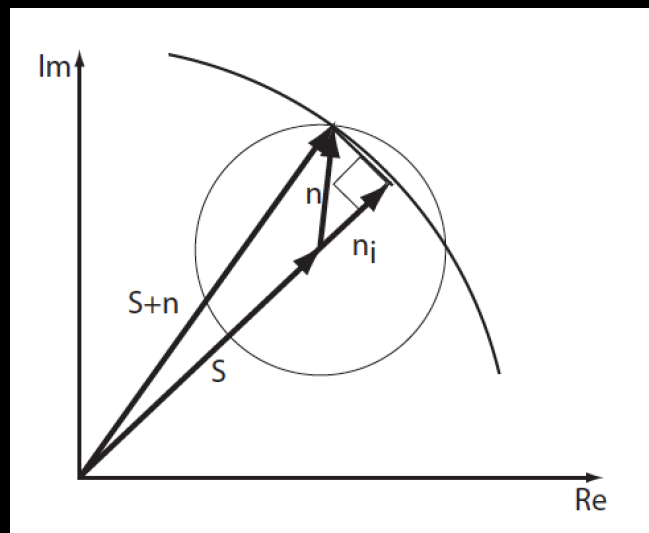


Phase Correction



Noise Consideration

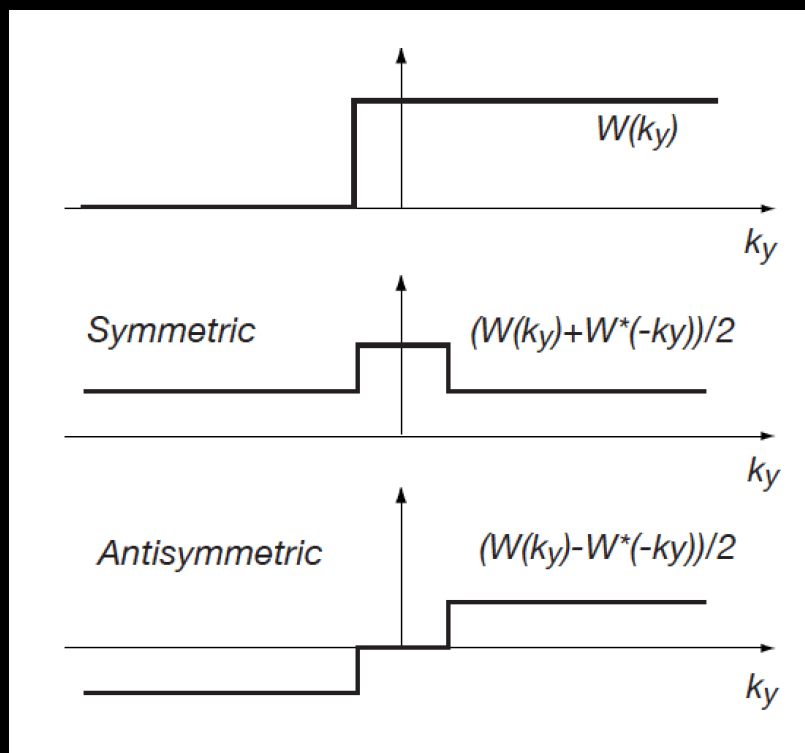
- Background in the phase corrected image has lower noise because one component of the complex noise has been suppressed



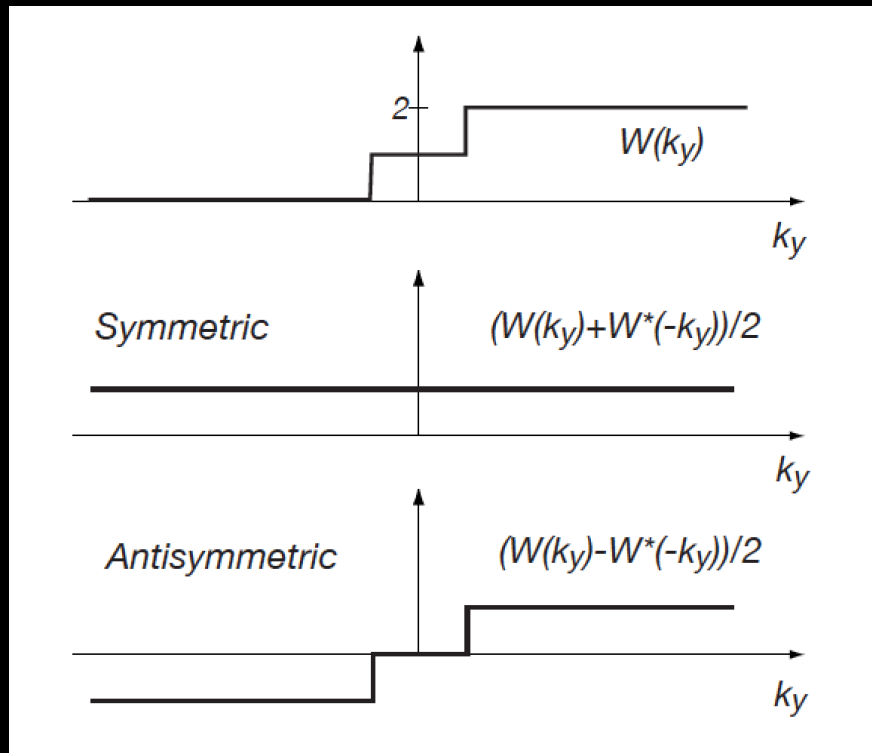
Homodyne Reconstruction

- Real part of an image corresponds to the conjugate symmetric component of the transform
- Imaginary part of an image corresponds to the conjugate anti-symmetric component of the transform

Symmetric and Antisymmetric Components



Symmetric and Antisymmetric Components

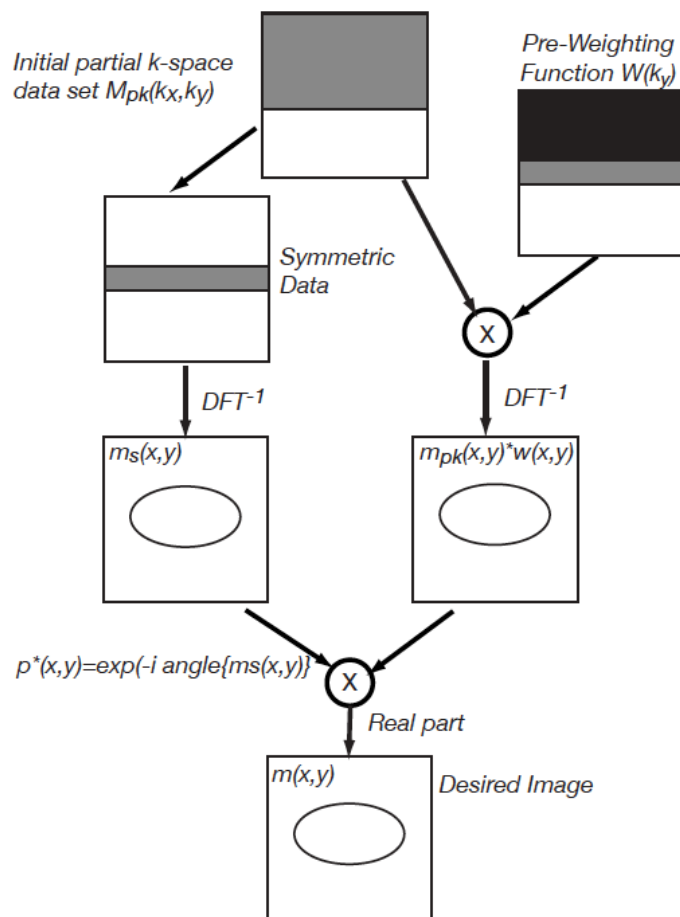
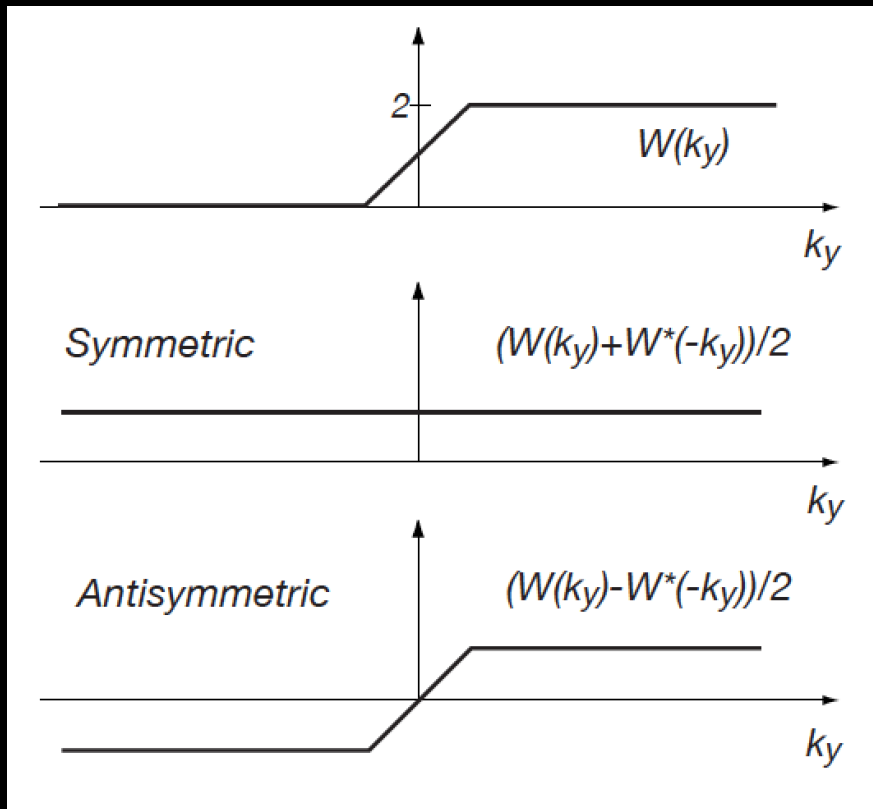


Weighting Function

$$m(x, y) = \text{Re}\{p^*(x, y)(m(x, y) * w(x, y))\}$$

- The phase correction in image space corresponds to a convolution in k-space
- The weighting sharp discontinuities of the weighting function can produce image artifacts

Preferred Weighting Function



MATLAB Code

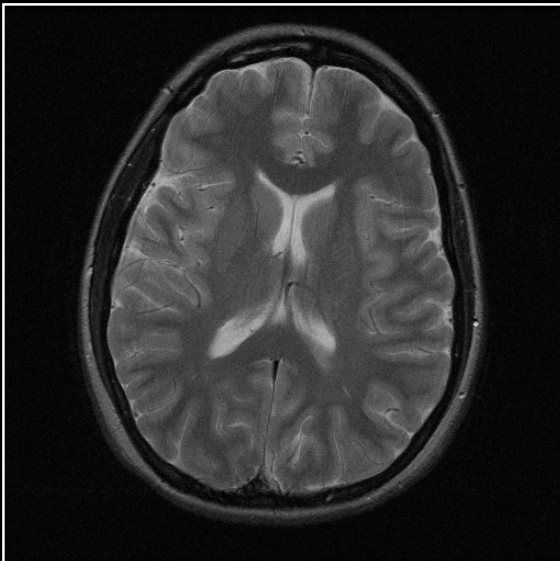
```
% Generate pre-weighting function W(ky)
Wld = zeros(nx,1);
Wld(1:hnover) = 2;
Wld(hnover+1:nx-hnover) = 2*(nx-2*hnover-1:-1:0)/(nx-2*hnover);
Wky = repmat(Wld,[1 nx]);

data_pw = data_pk.*Wky;
im_ph = fftshift(ifftn(fftshift(data_pw)));

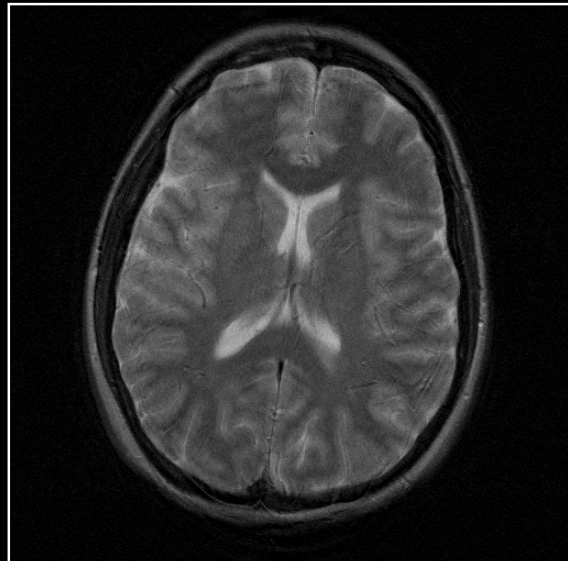
im_homodyne = im_ph.*exp(-1i*angle(im_ph));
```

Homodyne Reconstruction

Original

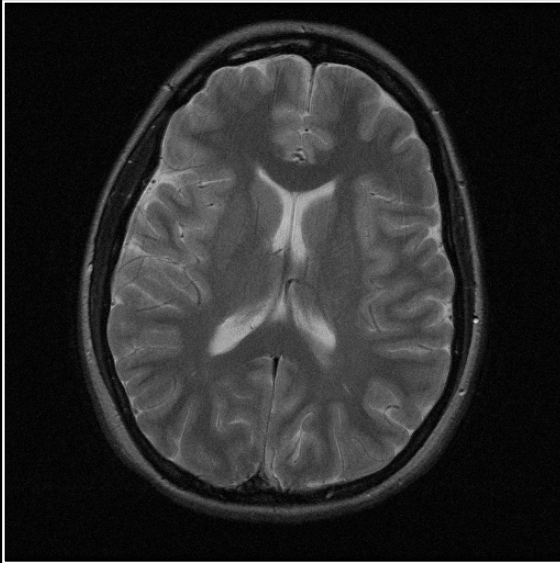


Phase Correction

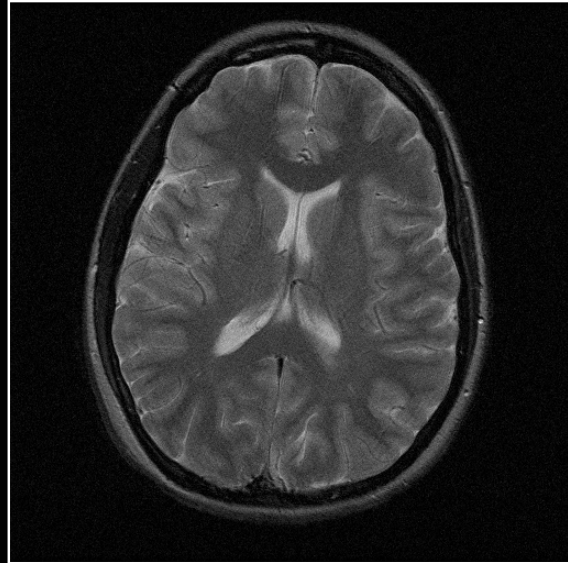


Homodyne Reconstruction

Original



Homodyne Recon



Summary of Direct Methods

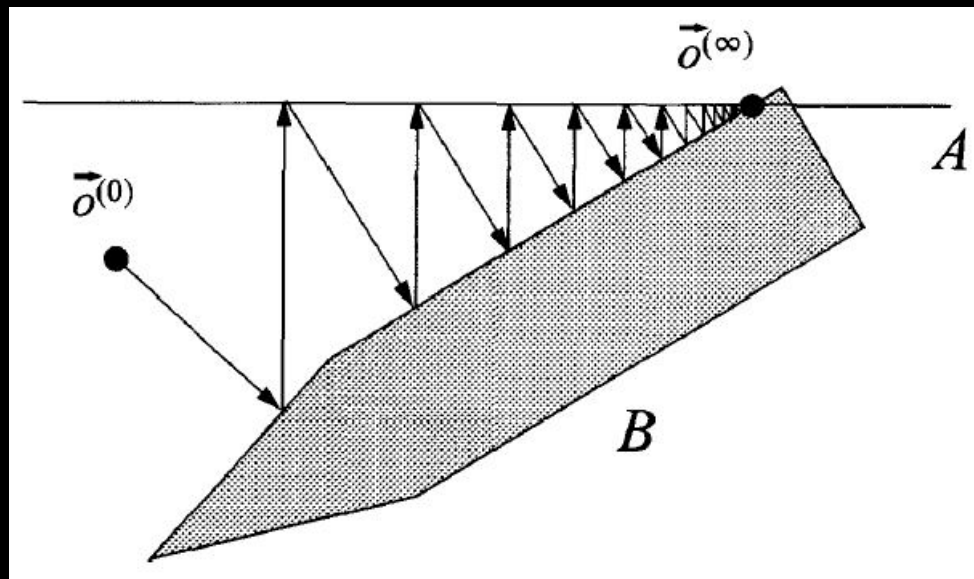
- Both homodyne and phase corrected conjugate synthesis approaches work well if image phase does not vary rapidly
- Problems with homodyne approach are the result of performing phase correction after conjugate synthesis.
- Problems with phase corrected conjugate synthesis approach are due to performing the conjugate synthesis after the phase correction

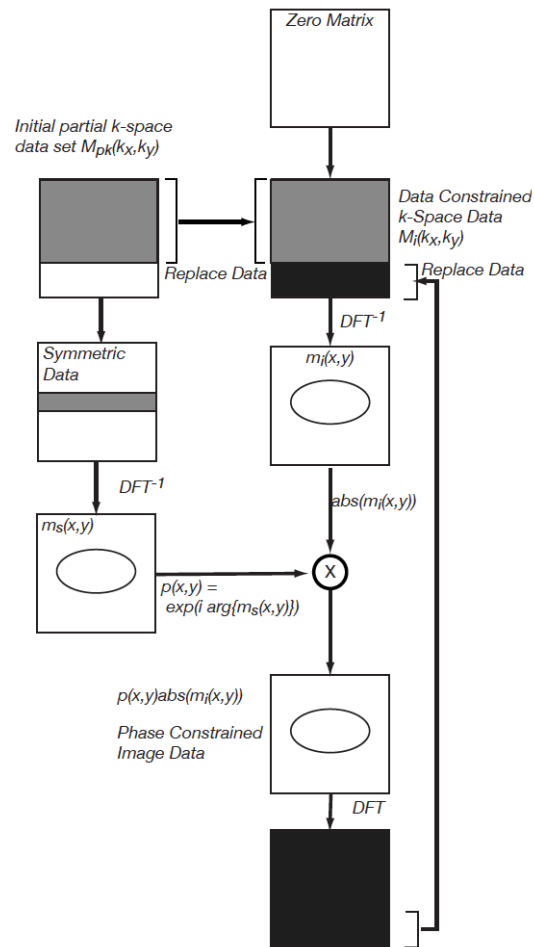
Iterative Reconstruction

$$m_i(x, y) = |m_i(x, y)|p(x, y)$$

- Estimate the missing k-space data by iteratively applying phase correction and conjugate synthesis
- In the image domain, the image phase is constrained to be that of the low resolution estimate
- In the frequency domain, the k-space data is constrained to match the acquired data when available

Projection Onto Convex Set (POCS)





MATLAB Code

```

threshold_pocs = 0.001;

% Zero padding for initial guess
im_init = fftshift(iffn(fftshift(data_pk))); % Inverse DFT
% Take only magnitude term & Apply phase term
im_init = abs(im_init).*exp(1i*angle(im_ph));

% FFT
tmp_k = fftshift(fftn(fftshift(im_init)));
diff_im = threshold_pocs + 1;

while (abs(diff_im) > threshold_pocs)
    tmp_k(1:nx-hnover,:) = data_pk(1:nx-hnover,:);
    tmp_im = fftshift(iffn(fftshift(tmp_k))); % Inverse DFT

    % Take only magnitude term & Apply phase term
    tmp_im = abs(tmp_im).*exp(1i*angle(im_ph));
    tmp_k = fftshift(fftn(fftshift(tmp_im)));

    % Compare the reconstructed image
    diff_im = abs(tmp_im - im_init);
    diff_im = sum(diff_im(:).^2);
    fprintf('Difference is %f\n',diff_im);

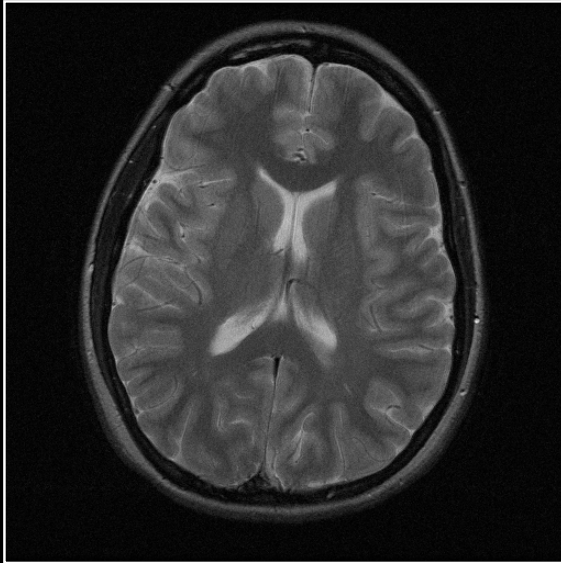
    im_init = tmp_im;
end

im_pocs = tmp_im;

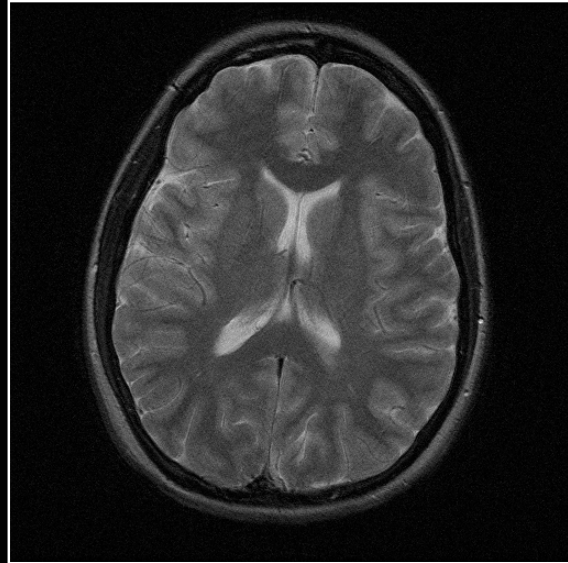
```

POCS Reconstruction

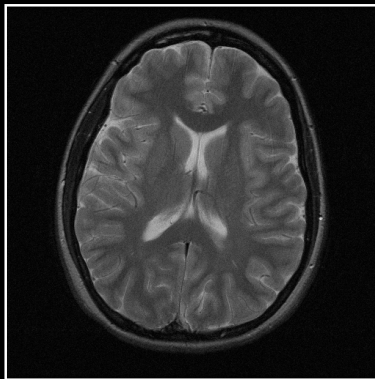
Original



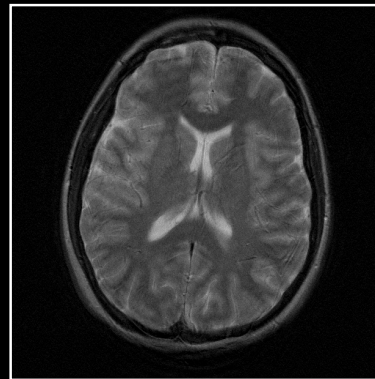
POCS Recon



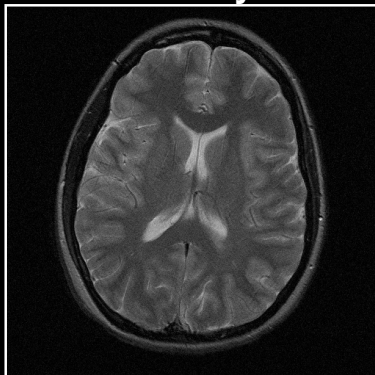
Original



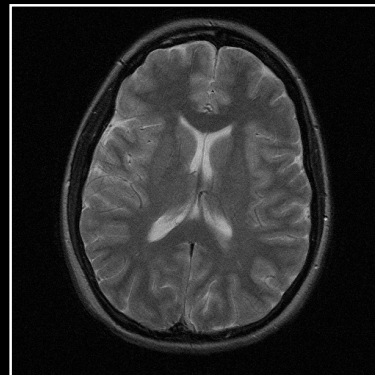
Phase Correction



Homodyne



POCS



Conclusions

- All of these algorithms work well when the image phase variations are smooth
- When the image phase changes rapidly, the homodyne algorithm produces ghosting
- POCS algorithm performs somewhat better as the k-space fraction decreases

Thanks!

- Next time:
 - Parallel Imaging
 - Read "Parallel Imaging Reconstruction" p522-544

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