Bulk Magnetization and Nuclear Precession

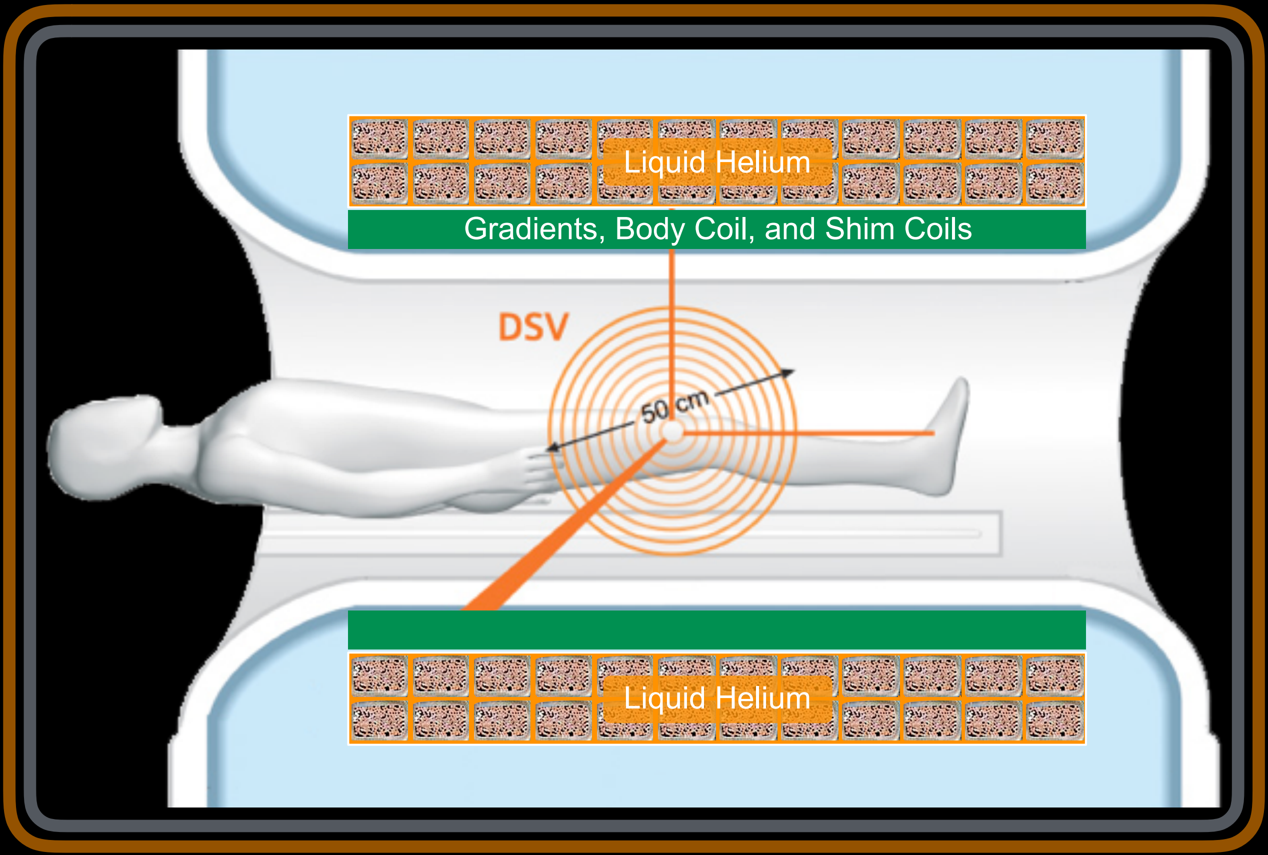
Lecture #2 – January 10th, 2018

Class Business

- **Matlab available via SEASNET**
  - http://www.seas.ucla.edu/acctapp
- **Website up and running**
  - http://mrrl.ucla.edu/education/m219/
  - Slides, video, code, reading, PDFs, etc.
  - Code available on website
    - Review code as needed
- **Meet with TAs for Matlab help.**

Lecture 1 - Summary

MRI uses a superconducting electromagnet!



Copper RF Shielding
Steel Magnetic Shielding

$$B = \mu I N L^{-1}$$

$1.5T = 4\pi \times 10^{-7} \cdot$ **508 A** $\cdot 235 \cdot 1\ m^{-1}$

$$\vec{B}_0 = B_0 \vec{k}$$

Homogeneity – <4ppm peak-peak variation (6μT @ 1.5T!)

Questions?

Bulk Magnetization and Nuclear Precession

Lecture #2 – January 10th, 2018

Lecture #2 Learning Objectives

- **Write down three equations describing the B_0 principles.**
- **Explain the importance of Zeeman splitting.**
- **Describe the importance of spin, charge, and mass to NMR.**
- **Write down the equation of motion for an ensemble of spins.**
- **Differentiate between free and forced precession in the laboratory and rotating frames.**
- **Solve for the bulk magnetization dynamics during free precession in the laboratory frame without relaxation.**

Main Field (B_0) - Principles

Dipoles to Images

$\vec{\mu}$	Magnetic Moment		
$\downarrow$		}	B_0
$\vec{M}$	Bulk Magnetization		
$\downarrow$		}	B_1
$\vec{M}_{xy}(t)$	Transverse Magnetization		
$\downarrow$		}	$G_{Encoding}$
$\vec{M}_{xy}(\vec{r},t)$	Spatially Encoded Magnetization		
$\downarrow$		}	Coil
$V(t)$	Received Voltage		
$\downarrow$		}	PSD
$S(t)$	Complex Signal		
$\downarrow$		}	$G_{Decoding}$
$S\left(\vec{k}\right)$	k-space signal		
$\downarrow$		}	Reconstruction
$I(\vec{r})$	Image		

Main Field (B_0) - Principles

- B_0 is a strong magnetic field
  - Polarizer
  - >1.5T
  - Z-oriented

$$\vec{B}_0 = B_0\vec{k}$$ Eqn. 3.5

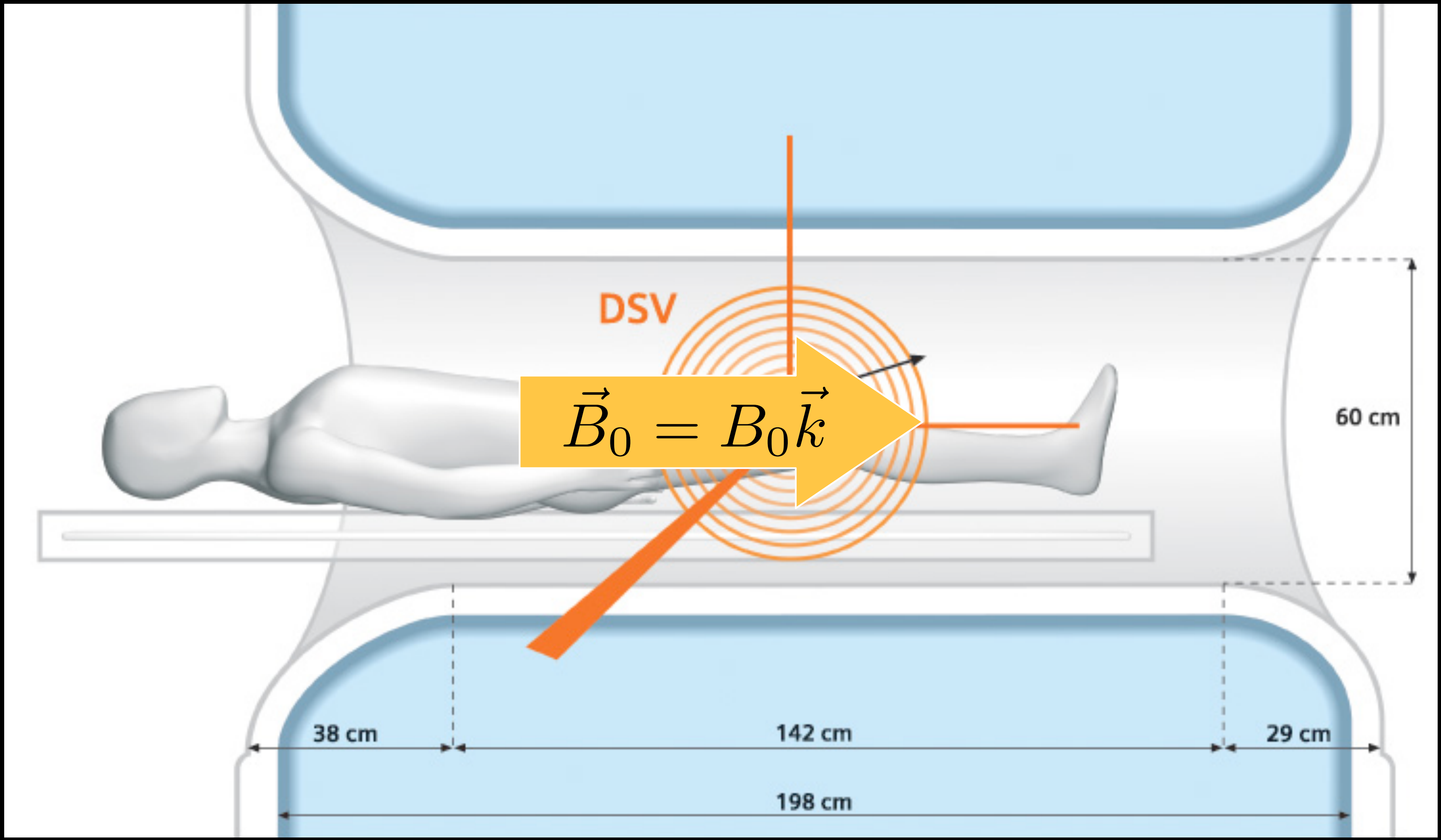
- B_0 generates bulk magnetization ($\vec{M}$)
  - More B_0, more

$$\vec{M} = \sum_{n=1}^{N_{total}} \vec{\mu}_n$$ Eqn. 3.26

- B_0 forces $\vec{M}$ to precess
  - Larmor Equation

$$\omega = \gamma B$$ Eqn. 3.18

B_0 Field



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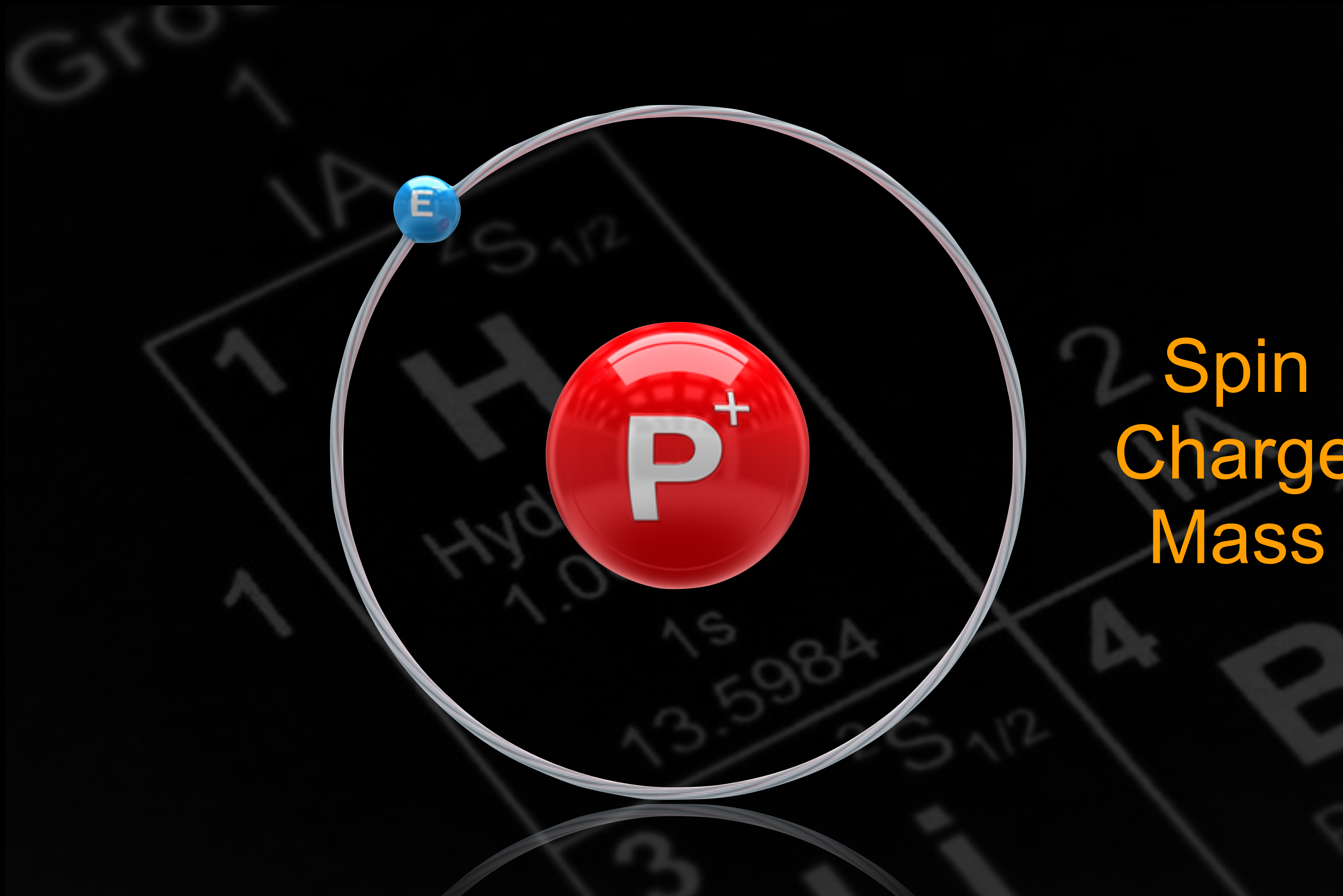
Hydrogen



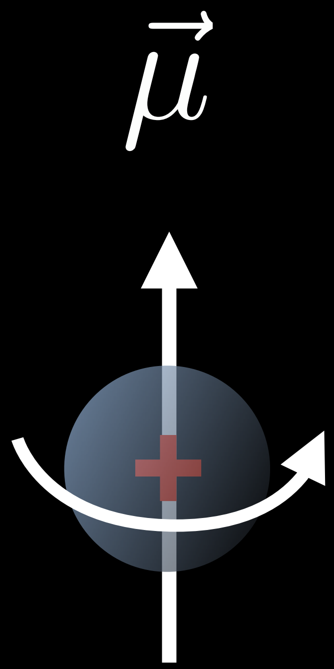
Spin
Charge
Mass

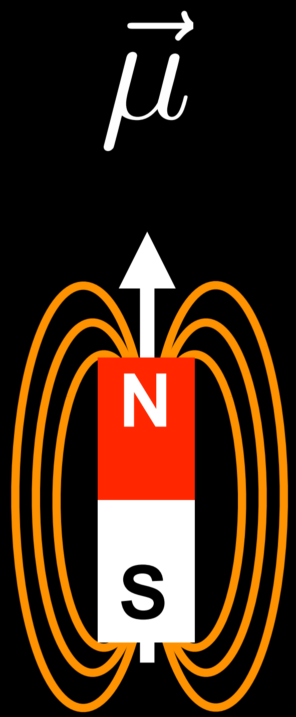
Hydrogen nuclei behave like magnetic dipoles.

Magnetic Dipole Moments

Spin + Charge ➟ Magnetic Moment ➟ $\vec{\mu}$ [J•T^{-1} *or* kg•m^2/s^2]

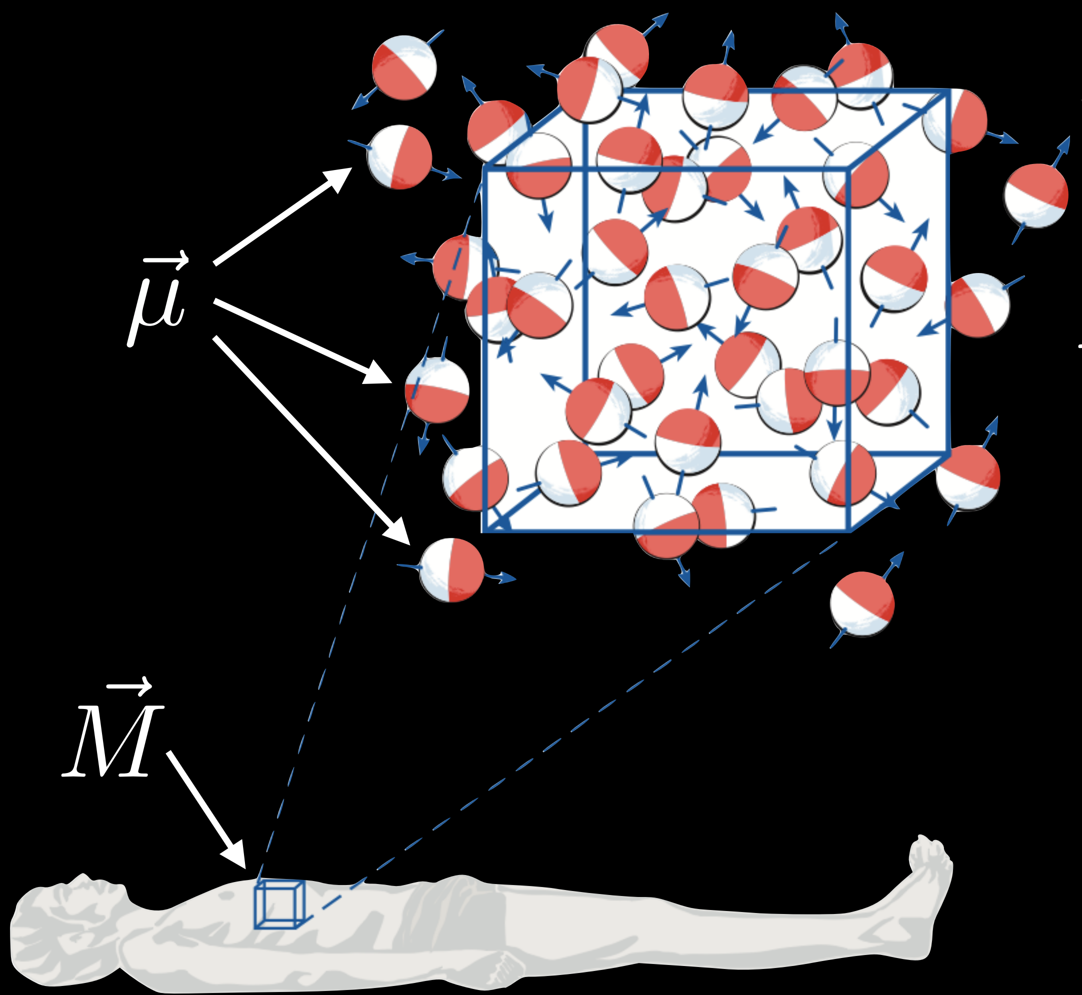
“a measure of the strength of the system's net magnetic source”
--http://en.wikipedia.org/wiki/Magnetic_moment





Hydrogen nuclei have magnetic dipole moments.

Bulk Magnetization



$$\vec{M} = \sum_{n=1}^{N_{total}} \vec{\mu}_n$$ Eqn. 3.26

$N_{total}=0.24\times10^{23}$ spins in a 2x2x10mm voxel

But not all spins contribute to our measured signal...

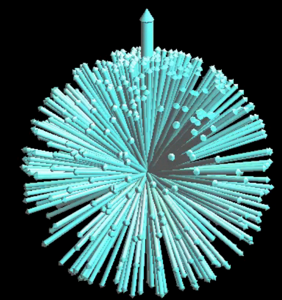
Equilibrium Bulk Magnetization

$$\vec{M} = \sum_{n=1}^{N_{total}} \vec{\mu}_n$$ Eqn. 3.26

$$\vec{M} = M_x\hat{i} + M_y\hat{j} + M_z\hat{k}$$ Eqn. 3.36

$$\vec{M}_z^0 = |\vec{M}| = \frac{\gamma^2\hbar^2 B_0 N_s}{4KT_s}$$ Eqn. 3.39

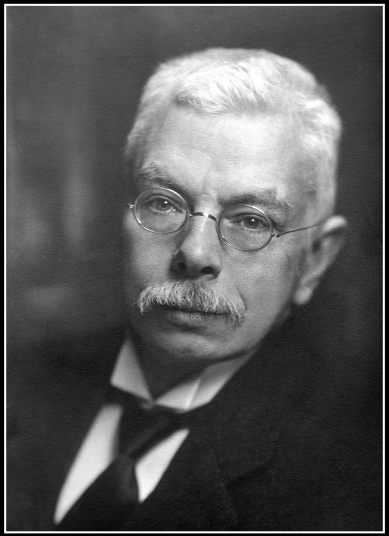
$$\vec{M}_x^0 = \vec{M}_y^0 = 0$$



Hanson, L. G. (2008). Concepts in MR Part A 32(A): 329-340.

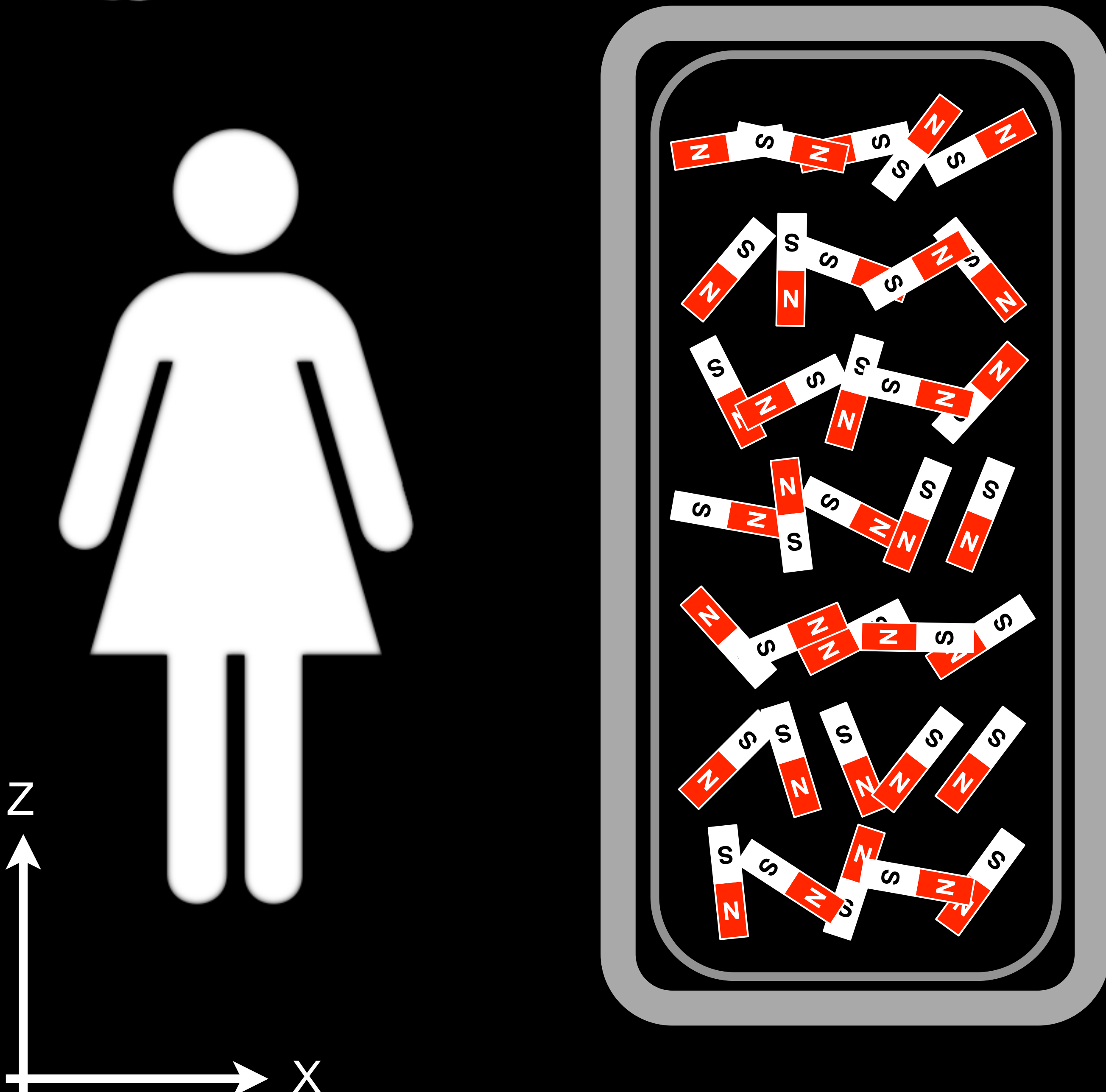
Bulk magnetization at equilibrium in a B_0 field.

Zeeman Splitting



Pieter Zeeman
b. 25 May 1865
d. 9 Oct 1943

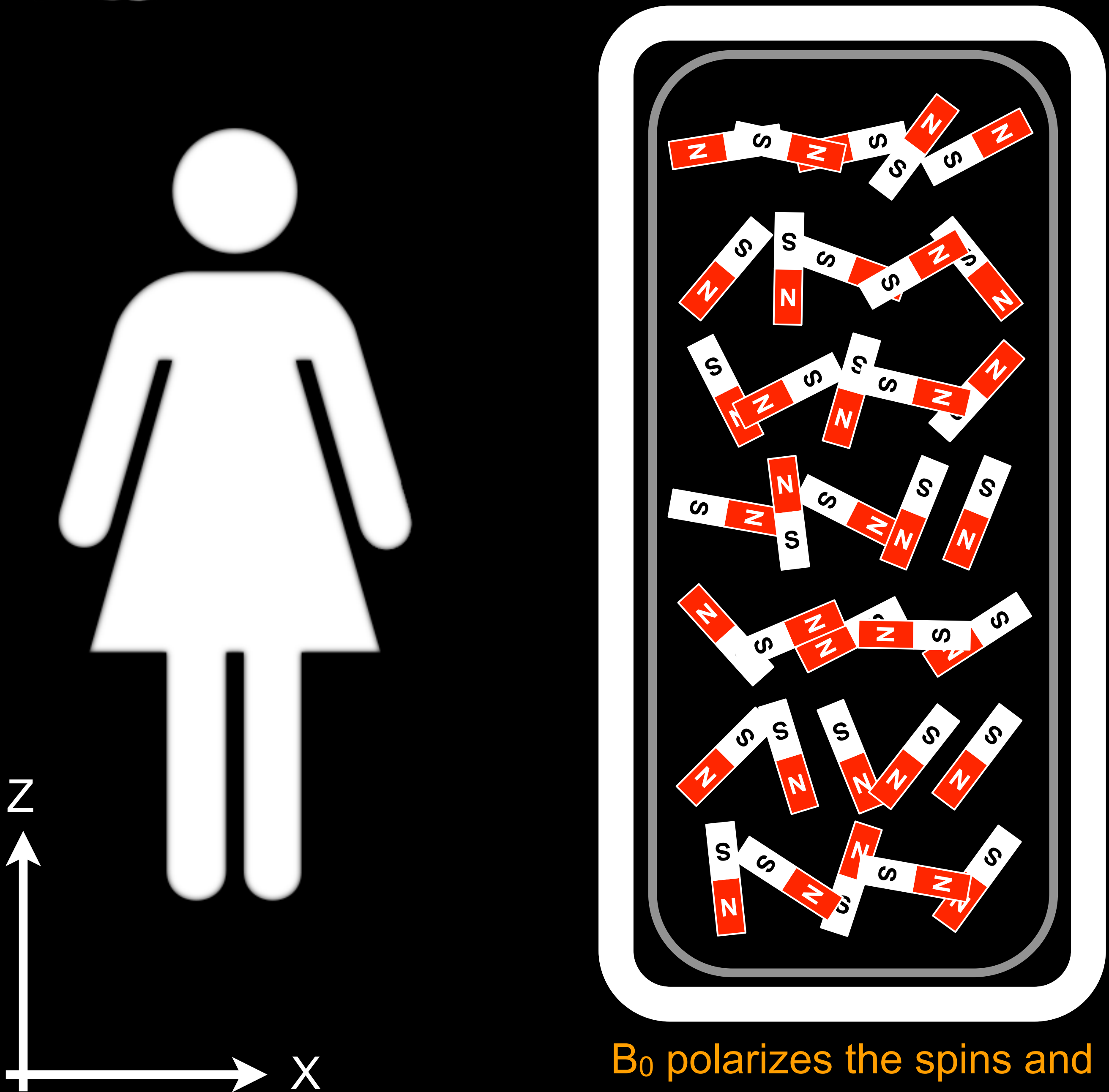
B_0 Field OFF



$$\vec{M} = \sum_{n=1}^{N_{total}} \vec{\mu}_n = 0$$

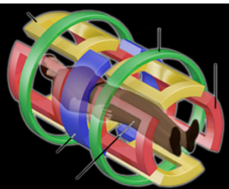
Spins point in all directions.

B_0 Field ON

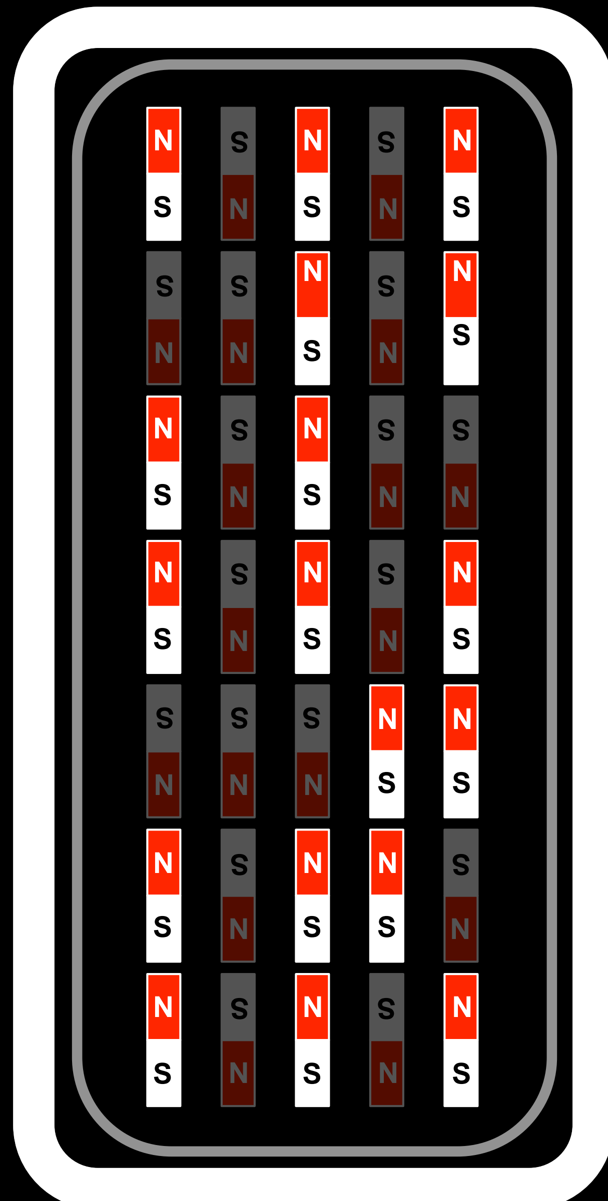


$$\vec{M} = \sum_{n=1}^{N_{total}} \vec{\mu}_n = M_z$$

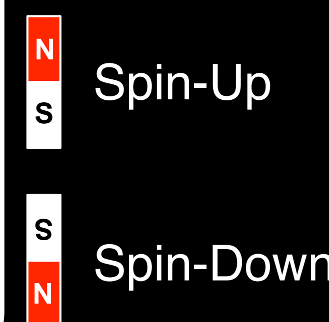
B_0 polarizes the spins and generates bulk magnetization.



B_0 Field ON

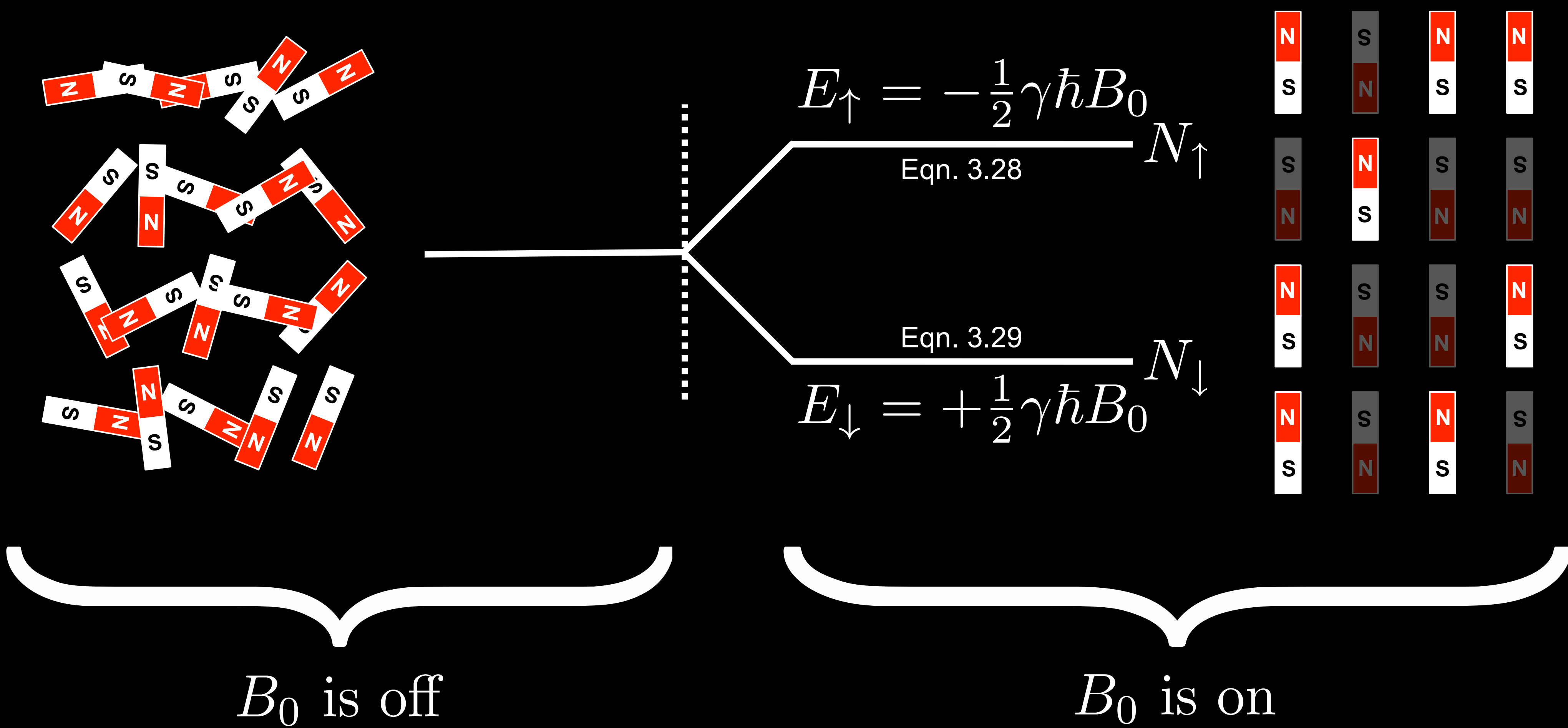


$$\vec{M} = \sum_{n=1}^{N_{total}} \vec{\mu}_n = M_z$$



Only a very small number are spin-up relative to spin-down.

Zeeman Splitting



$N_\uparrow$ = Spin-Up State, Low Energy

$N_\downarrow$ = Spin-Down State, High Energy

Zeeman Splitting

$$\frac{N_\uparrow - N_\downarrow}{N_{total}} \approx \frac{\gamma h B_0}{2KT}$$

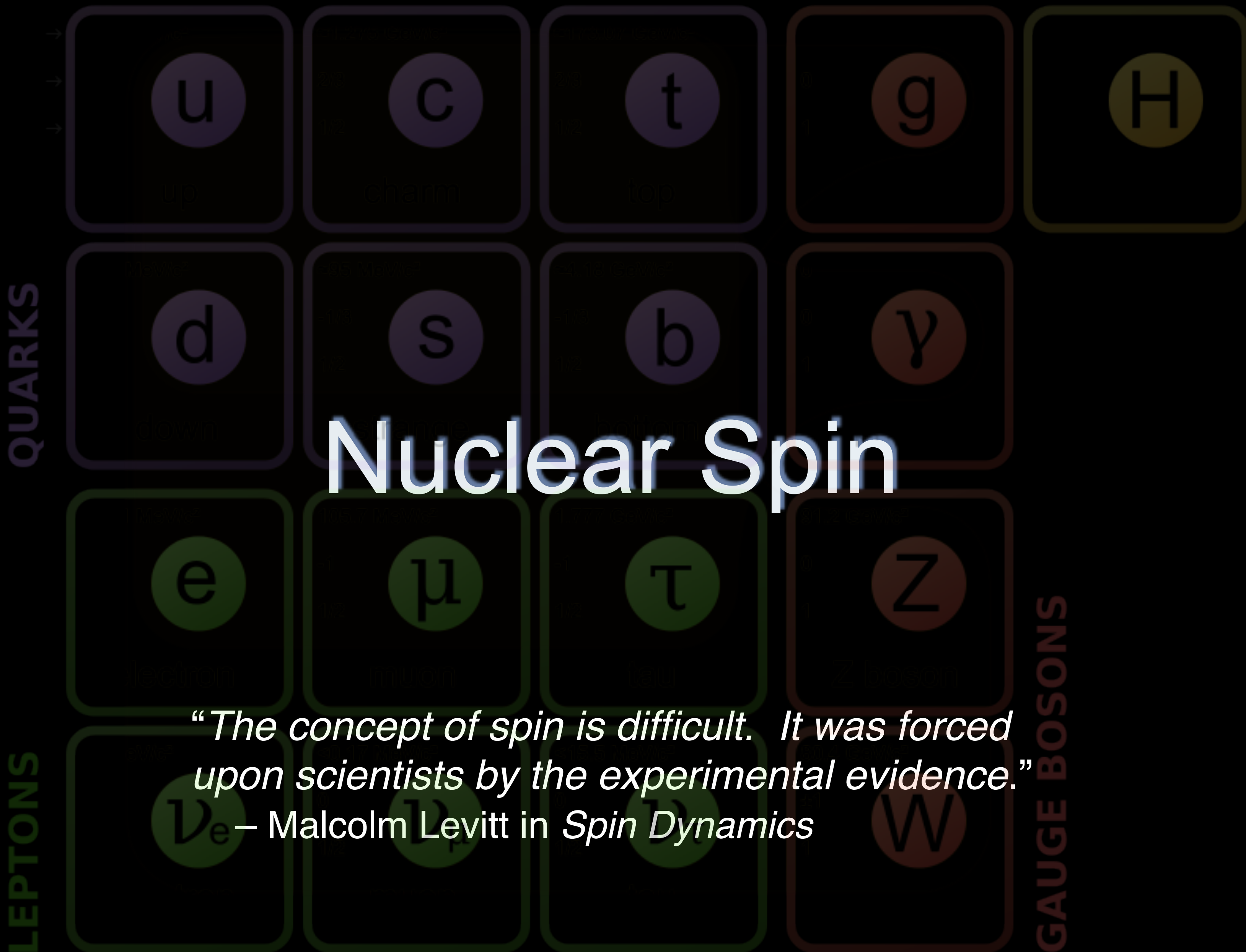
Eqn. 3.35

$$\begin{aligned}
\gamma &= 42.58 \times 10^6 \text{ Hz/T} \\
h &= 6.6 \times 10^-34 \text{ J} \cdot \text{s [Planck' Constant]} \\
T &= 300\text{K (room temperature)} \\
K &= 1.38 \times 10^{-23} \text{ J/K [Boltzmann Constant]} \\
B_0 &= 1.5\text{T}
\end{aligned}$$

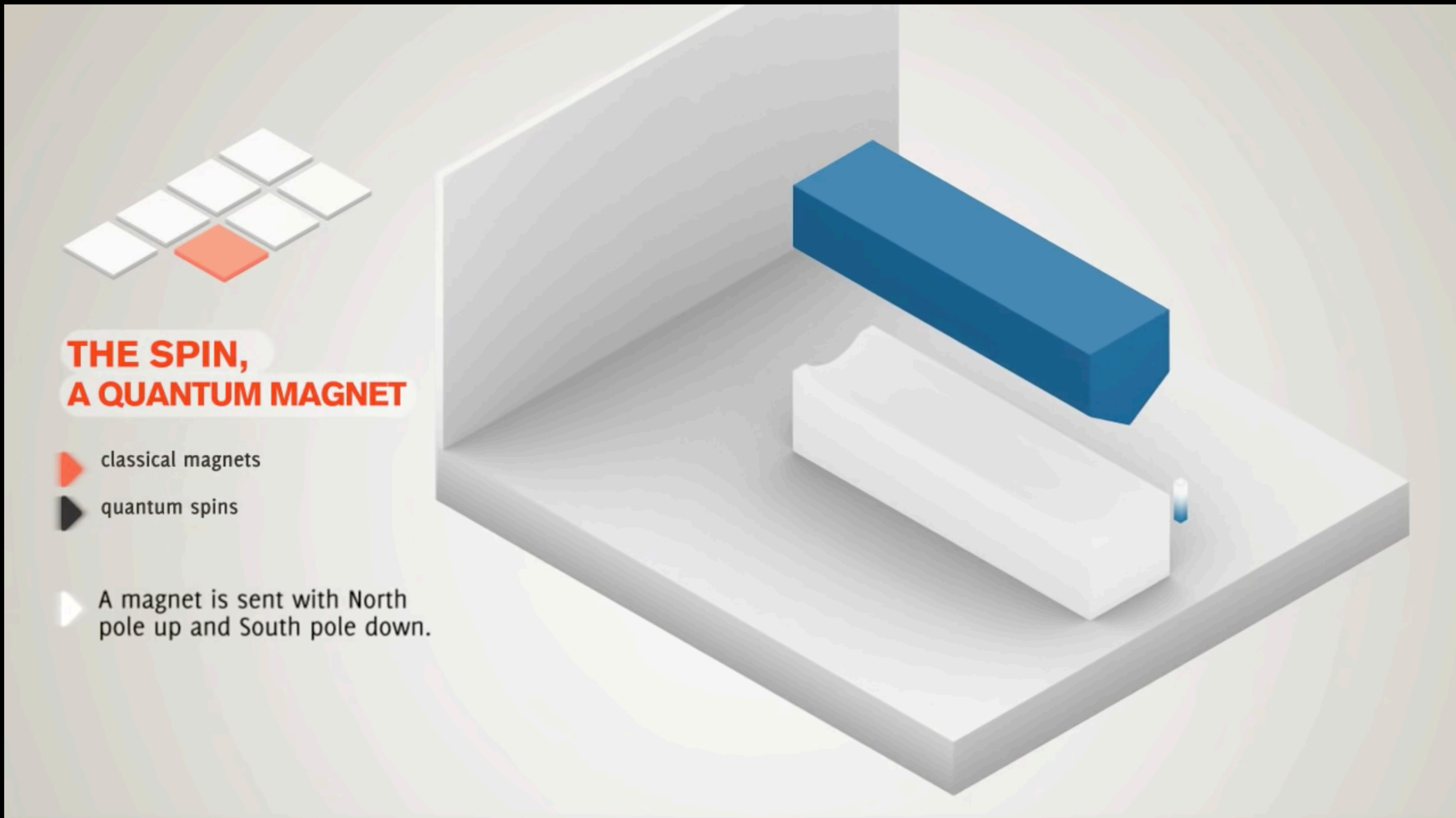
$$\frac{N_\uparrow - N_\downarrow}{N_{total}} \approx \frac{42.58 \times 10^6 \cdot 6.6 \times 10^{-34} \cdot 1.5}{2 \cdot 1.38 \times 10^{-23} \cdot 300} \approx 4.5 \times 10^{-6}$$

Nuclear Spin

"The concept of spin is difficult. It was forced upon scientists by the experimental evidence."
– Malcolm Levitt in *Spin Dynamics*

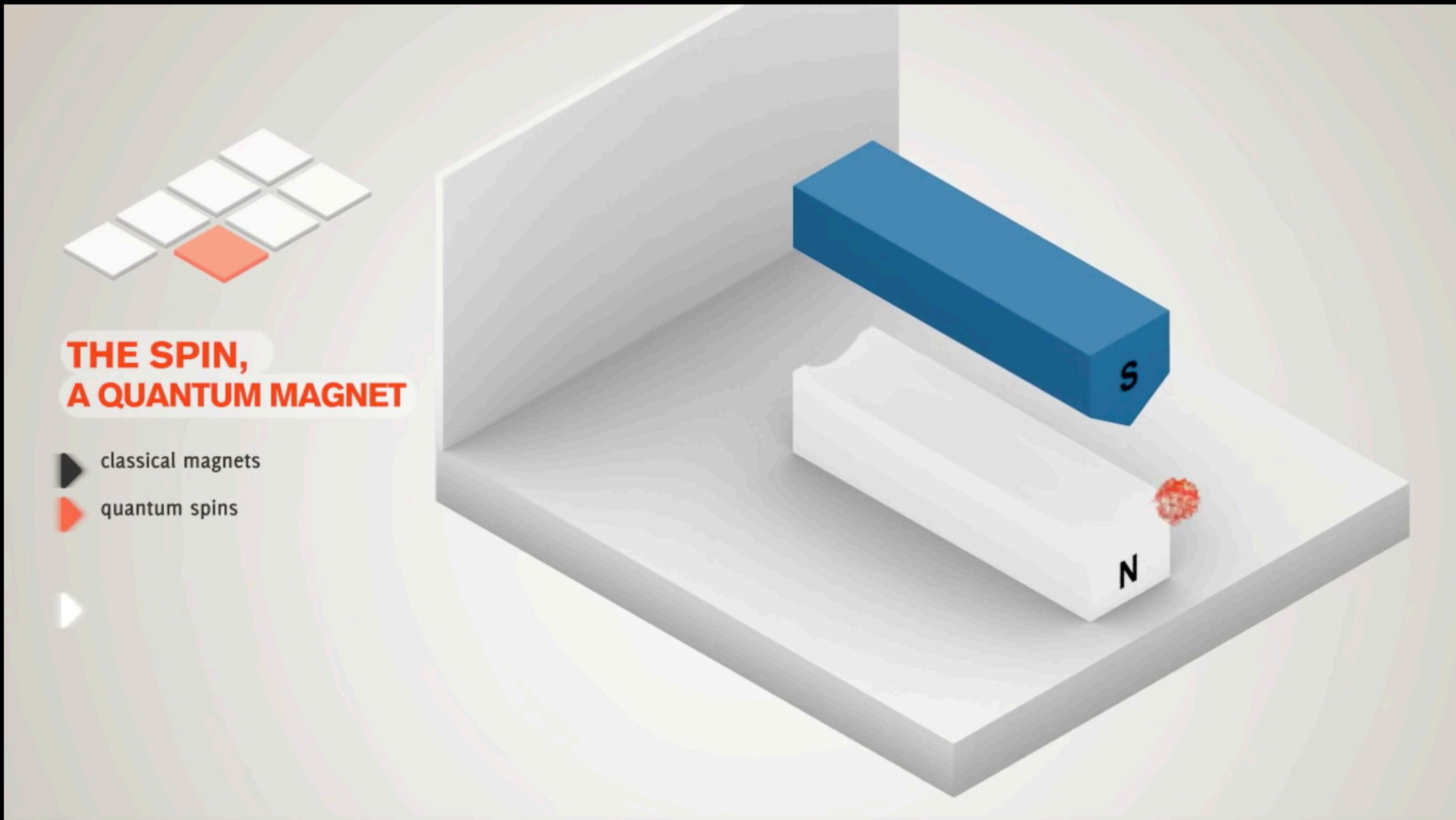


Quantum Spin Thought Experiment



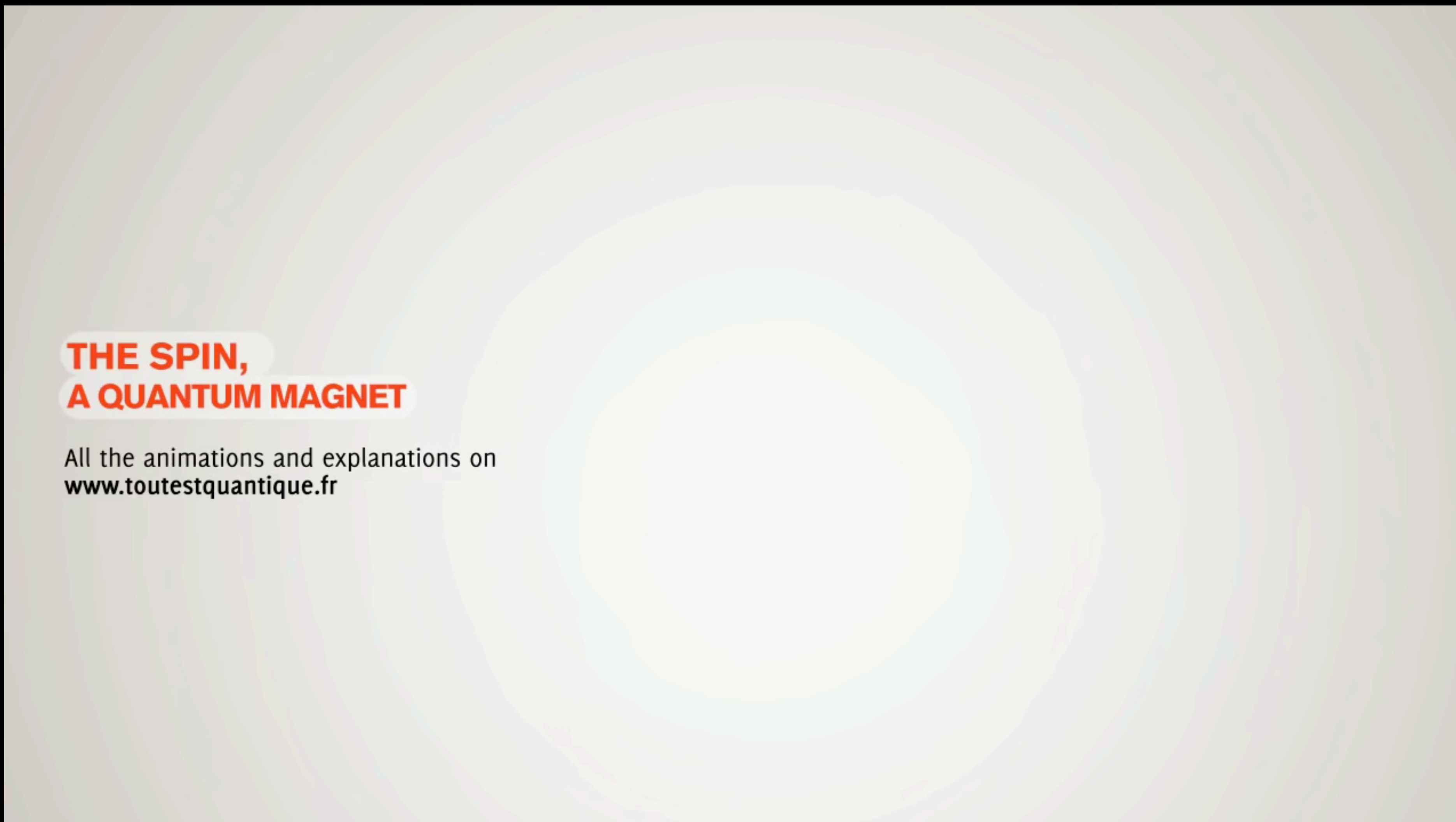
To where are *classical* magnets deflect, if sent in with a range of orientations?

Quantum Spin Thought Experiment



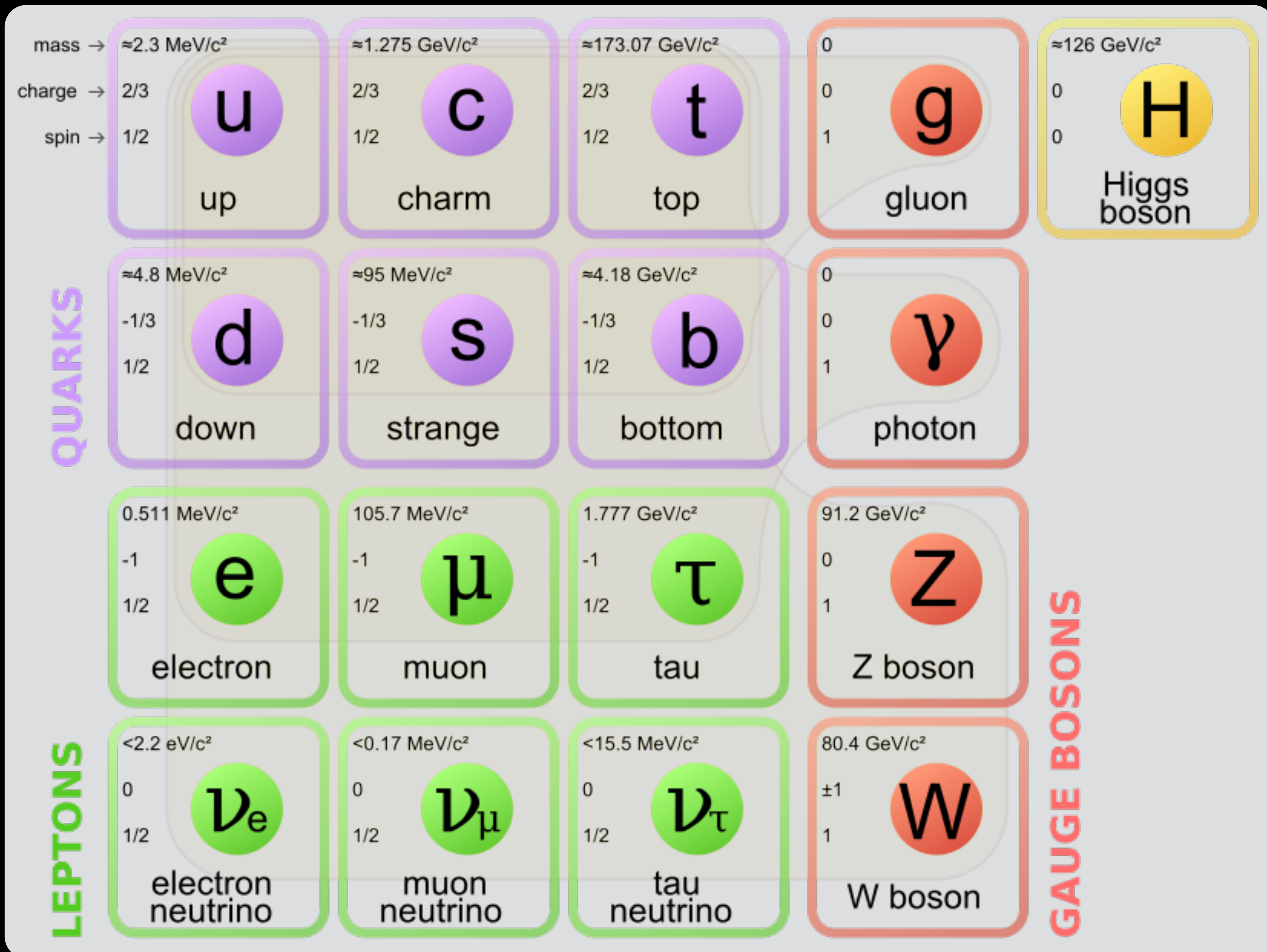
To where are *quantum spins* deflect, if sent in with a range of orientations?

How was spin first observed?



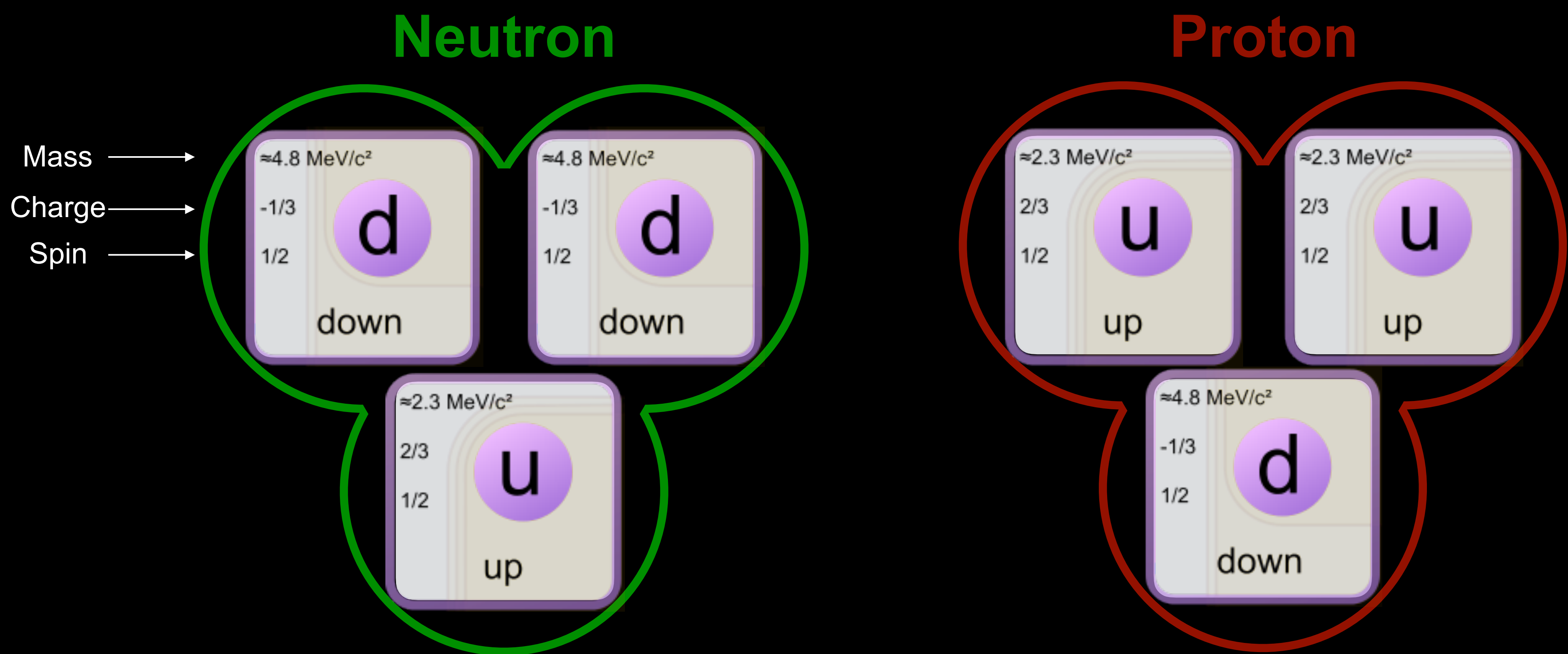
Otto Stern and Walther Gerlach performed the Stern–Gerlach experiment in Frankfurt, Germany in 1922.

The Standard Model



http://en.wikipedia.org/wiki/Standard_Model

Nuclear Spin - Quarks



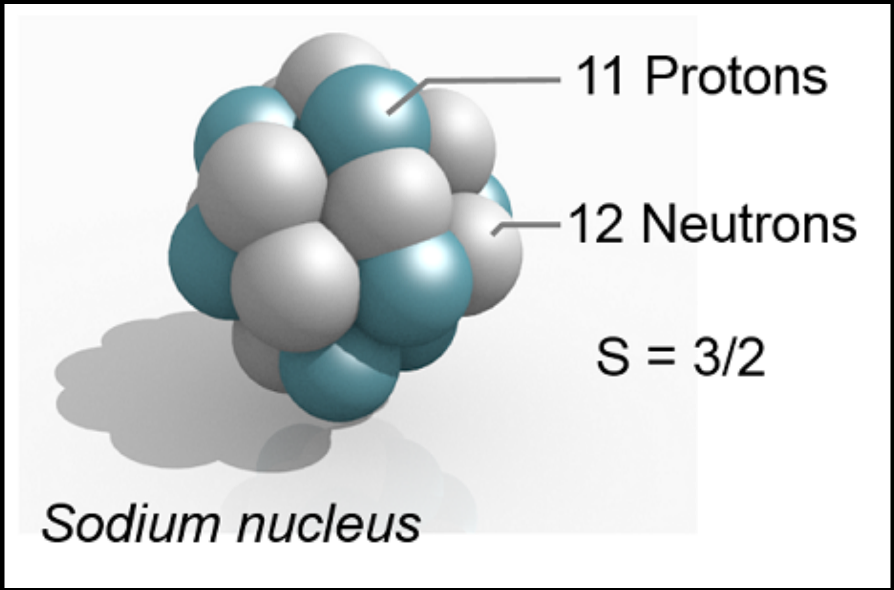
Charge=0
Spin=1/2

Charge=+e
Spin=1/2

Spin Crisis!

Nuclear Spin Quantum Number (I)

- A nucleus is NMR active only if $I \neq 0$
- ***Zero Spin*** – _Even mass number_ and _even charge number_
  - ^{12}C and ^{16}O
- ***Half-integral Spin*** – _Odd mass number_
  - Spin-1/2 – ^{1}H, ^{13}C, ^{15}N, ^{19}F, ^{31}P
  - Spin-3/2 – ^{23}Na
  - Spin-5/2 – ^{17}O
- ***Integral Spin*** – _Even mass number_ and _odd charge number_
  - ^{2}H and ^{14}N



NMR Active Nuclei

Isotope	*Spin [I]*	Natural Abundance	Gyromagnetic Ratio [MHz/T]	Relative Sensitivity	Absolute Sensitivity
^{1}H	1/2	0.9980	42.57	1	9.98E-01
^{2}H	1	0.0160	6.54	0.015	2.40E-04
^{12}C	0				---
^{13}C	1/2				1.76E-04
^{14}N	1				9.96E-04
^{15}N	1/2				4.00E-06
^{16}O	0				---
^{17}O	5/2				1.16E-05
^{19}F	1/2				8.30E-01
^{23}Na	3/2				9.30E-02
^{31}P	1/2				6.60E-02

```
% Filename: PAM_Lec02_Relative_Sensitivity.m
%
% Calculate the relative and absolute sensitivity.
%
% DBE@UCLA 2016.01.04

%%  Define some constants
GMR_1H=42.57e6;  % Gyromagnetic ratio for 1H [Hz/T]
GMR_2H= 6.54e6;  % Gyromagnetic ratio for 2H [Hz/T]

SPN_1H=0.5;        % Spin for 1H
SPN_2H=1.0;        % Spin for 2H

NA_1H=0.9980;     % Natural abundance of 1H
NA_2H=0.0160;     % Natural abundance of 2H

%% Calculate the "sensitivity"
S_1H=(GMR_1H*2*pi)^(11/4)*SPN_1H*(SPN_1H+1); % "Sensitivity" for 1H
S_2H=(GMR_2H*2*pi)^(11/4)*SPN_2H*(SPN_2H+1); % "Sensitivity" for 2H

%% Calculate the relative sensitivity
RS_2H=S_2H/S_1H

%% Calculate the absolute sensitivity
AS2H=(RS_2H*NA_2H)
```

The ***relative*** sensitivity is at constant magnetic field and equal number of nuclei.

– Using a factor of $\gamma^{\frac{11}{4}} I\left(I+1\right)$; ^{1}H is the reference standard.

The ***absolute*** sensitivity is the relative sensitivity multiplied by natural abundance.

NMR Active Nuclei

Isotope	*Spin [I]*	Natural Abundance	Gyromagnetic Ratio [MHz/T]	Relative Sensitivity	Absolute Sensitivity
^{1}H	1/2	0.9980	42.57	1	9.98E-01
^{2}H	1	0.0160	6.54	0.015	2.40E-04
^{12}C	0	0.9890	---	---	---
^{13}C	1/2	0.0110	10.71	0.016	1.76E-04
^{14}N	1	0.9960	3.08	0.001	9.96E-04
^{15}N	1/2	0.0040	-4.32	0.001	4.00E-06
^{16}O	0	0.9890	---	---	---
^{17}O	5/2	0.0004	-5.77	0.029	1.16E-05
^{19}F	1/2	1.0000	40.05	0.83	8.30E-01
^{23}Na	3/2	1.0000	11.26	0.093	9.30E-02
^{31}P	1/2	1.0000	17.24	0.066	6.60E-02

The ***relative*** sensitivity is at constant magnetic field and equal number of nuclei.

– Using a factor of $\gamma^{\frac{11}{4}} I\,(I+1)$; ^{1}H is the reference standard.

The ***absolute*** sensitivity is the relative sensitivity multiplied by natural abundance.

Gyromagnetic Ratio

- Gyromagnetic Ratio [MHz/T]
  - Physical constant
  - Unique for each NMR active nuclei
  - Ratio of the magnetic moment to the angular momentum

$$\vec{\mu} = \gamma \vec{S}$$

  - Measured empirically
  - Governs the frequency of ***precession***
  - Gamma vs. Gamma-bar

$$\bar{\gamma} = \gamma / 2\pi$$

[MHz/T]

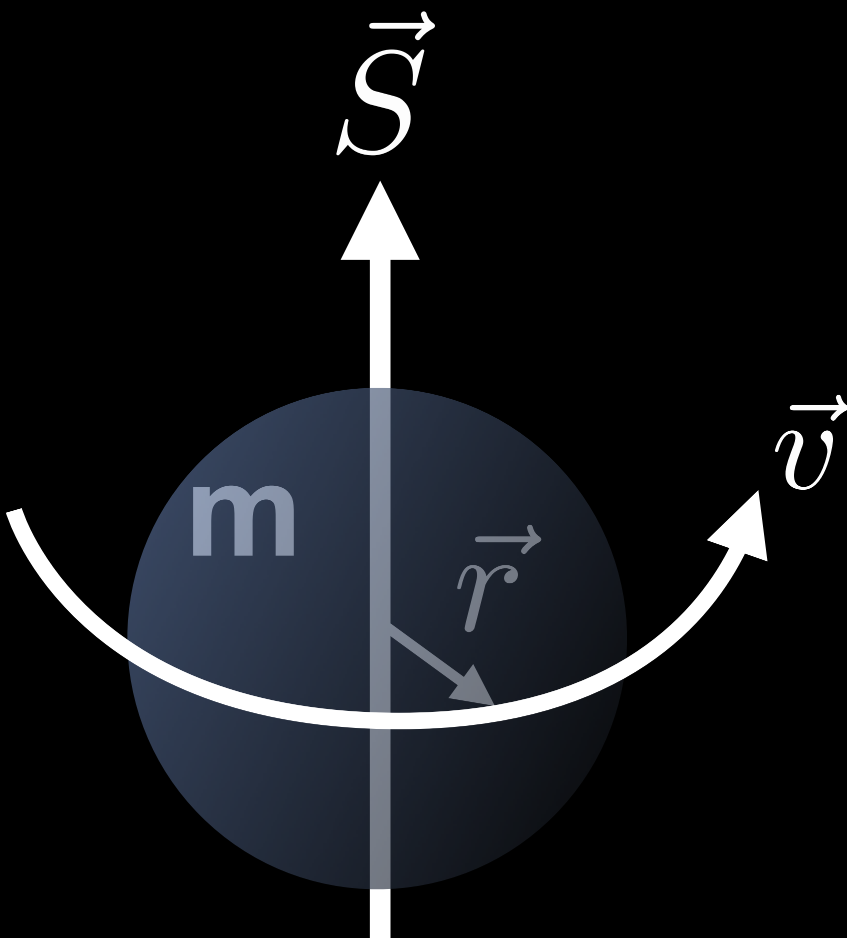
What are the implications of spin?

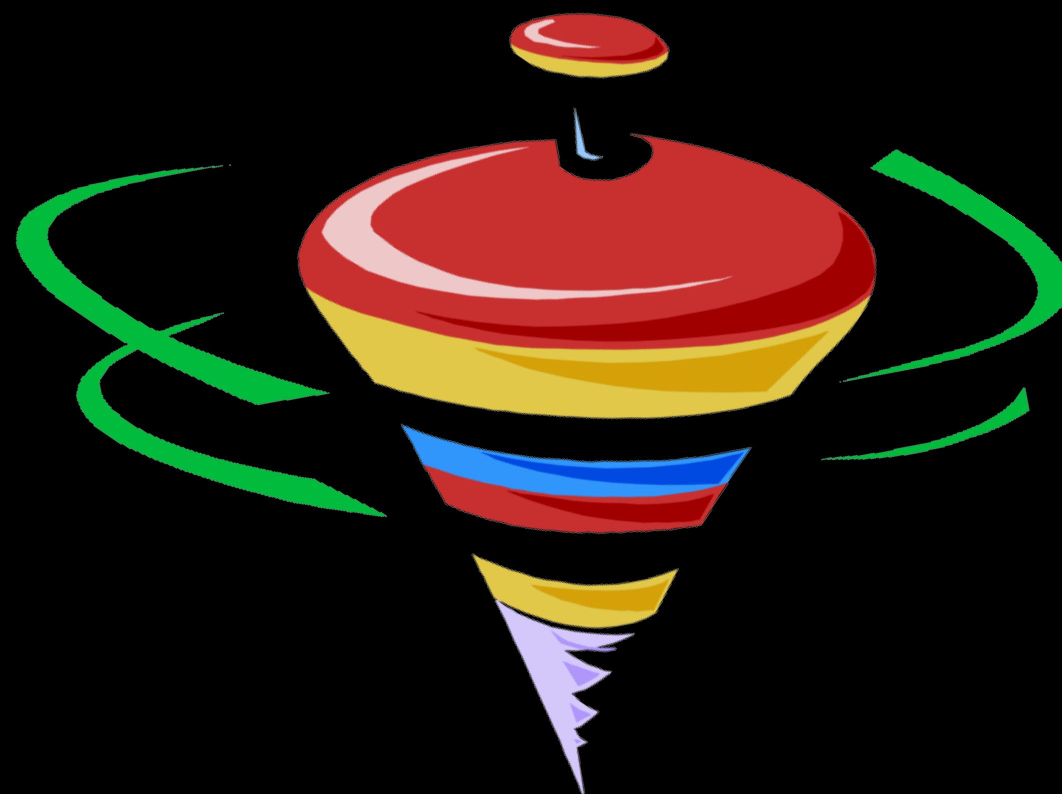
Nuclear Precession

Movie Courtesy of Donald Plewes @ U. Toronto

Spin Angular Momentum

Spin + Mass ➠ Spin Angular Momentum ➠ $\vec{S}$ [kg·m²s⁻¹]

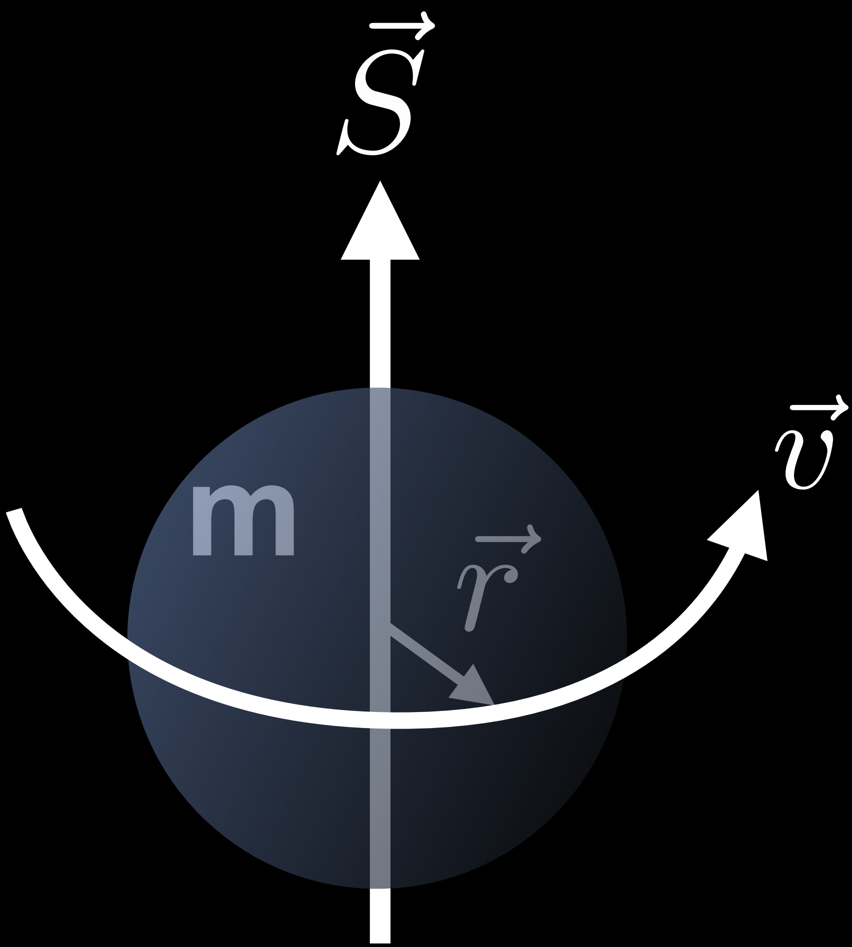




Hydrogen nuclei have spin angular momentum.

Spin Angular Momentum

Spin + Mass ➠ Spin Angular Momentum ➠ $\vec{S}$ [kg·m²s⁻¹]



$$\vec{S} = \vec{r} \times \vec{p}$$
$$= \vec{r} \times m\vec{v}$$

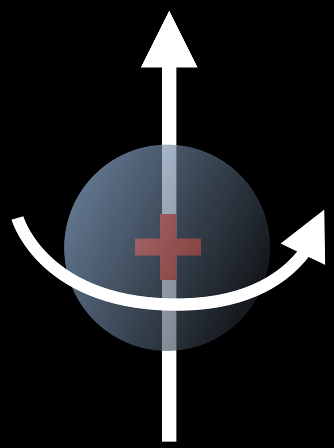
Hydrogen nuclei have spin angular momentum.

Magnetic Dipole Moments

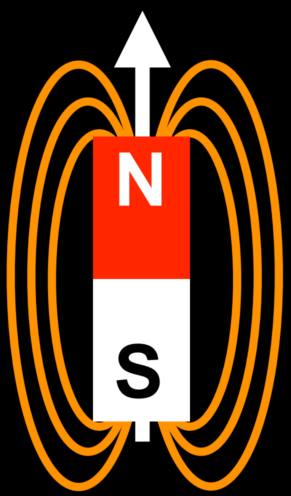
Spin + Charge ➟ Magnetic Moment ➟ $\vec{\mu}$ [J•T^{-1} *or* kg•m^2/s^2]

"a measure of the strength of the system's net magnetic source"
--http://en.wikipedia.org/wiki/Magnetic_moment

$\vec{\mu}$

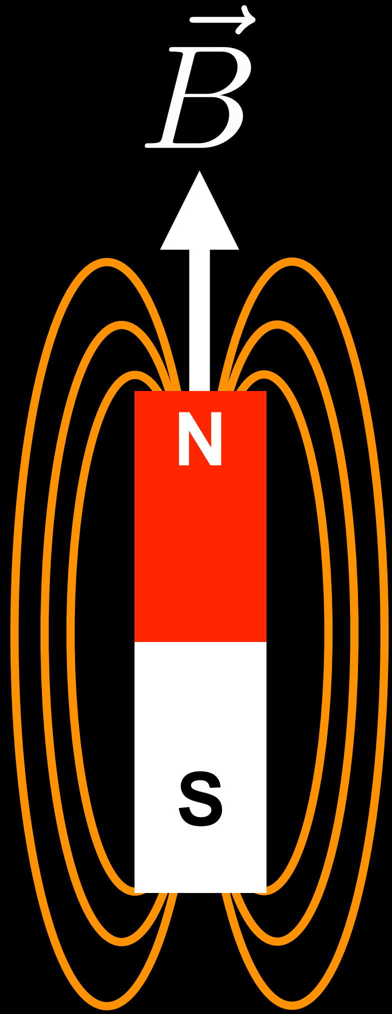


$\vec{\mu}$



Hydrogen nuclei have magnetic dipole moments.

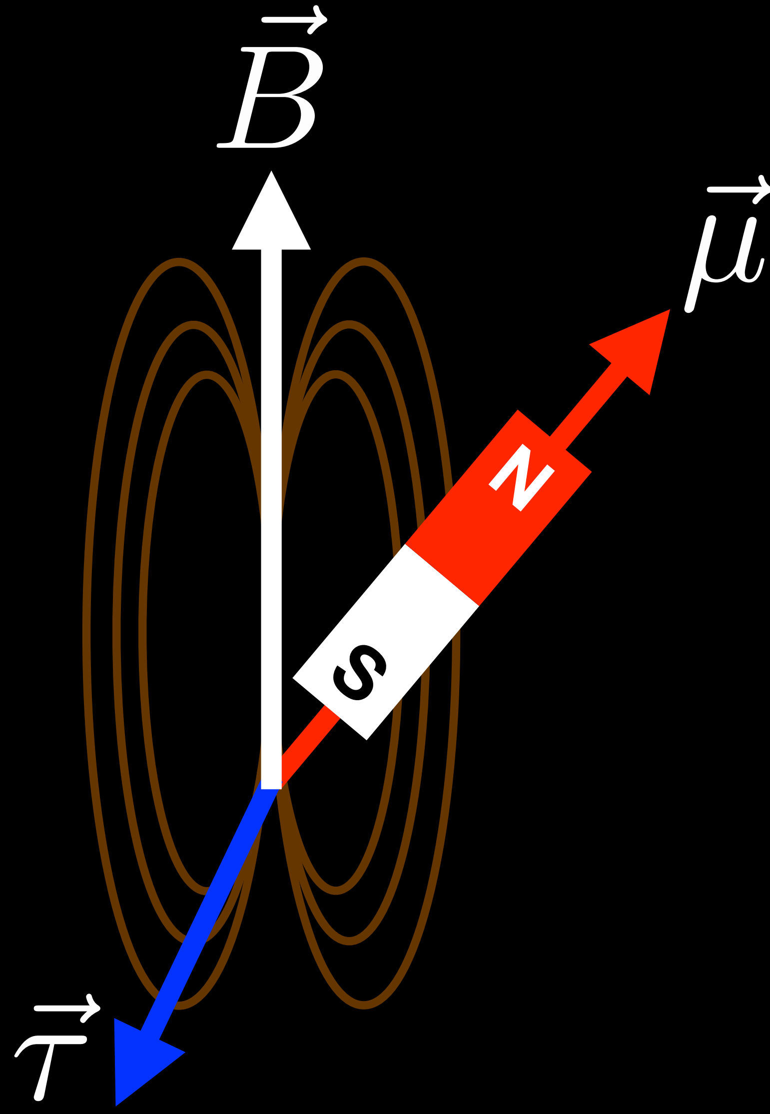
B-Field



"vector field which can exert a magnetic force on moving electric charges and on magnetic dipoles"
--http://en.wikipedia.org/wiki/Magnetic_field

Magnetic Dipole in a B-Field

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$



B_0 exerts a torque on the ^1H magnetic dipole moment.

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$$\omega = \gamma B$$ Eqn. 3.18

Spin vs. Precession

- **Spin**
  - Intrinsic form of angular momentum
  - Quantum mechanical phenomena
  - No classical physics counterpart
    - Except by hand-waving analogy…
- **Precession**
  - Spin+Mass+Charge give rise to precession

Spin vs. Precession

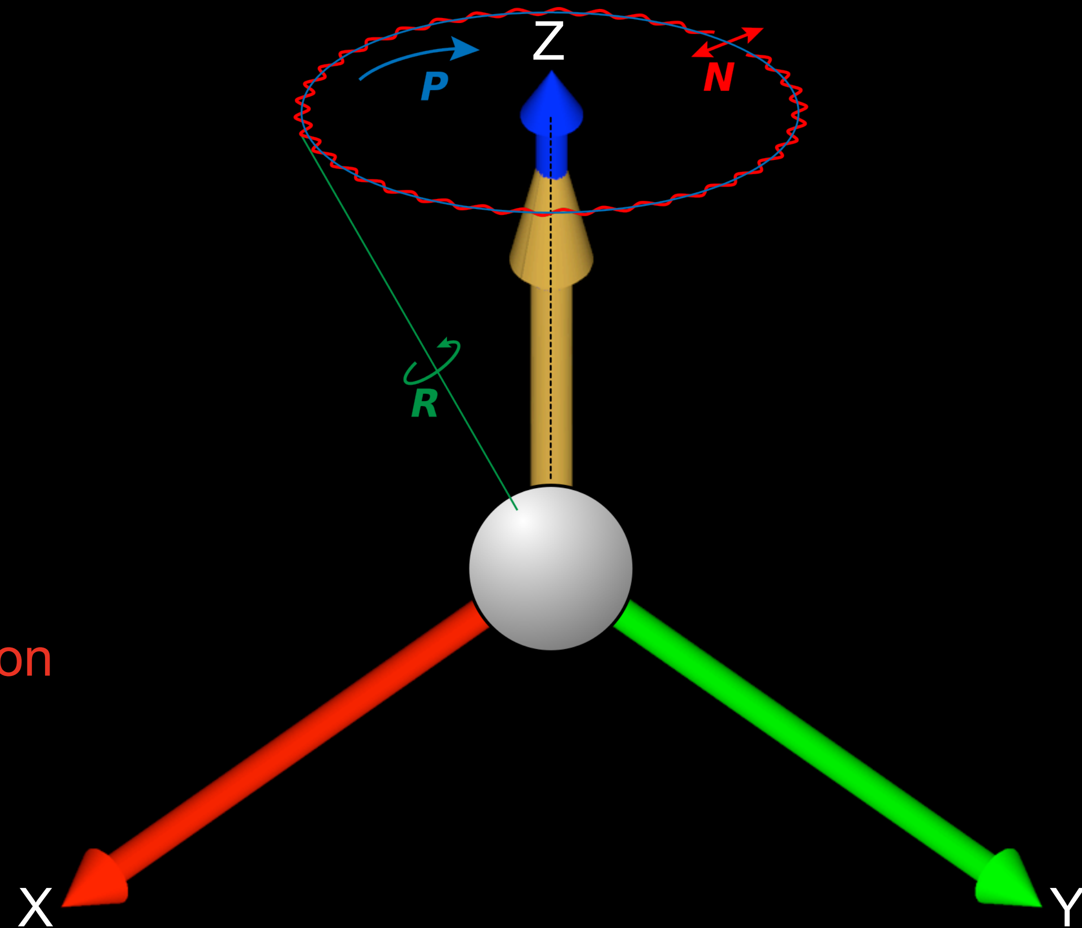
^{1}H has intrinsic Spin (***R***)

$\omega_0 = \gamma B_0$ Free Precession (***P***)

Combined with…

$\omega_1 = \gamma B_1$ Nutation (***N***)

– Forced Precession

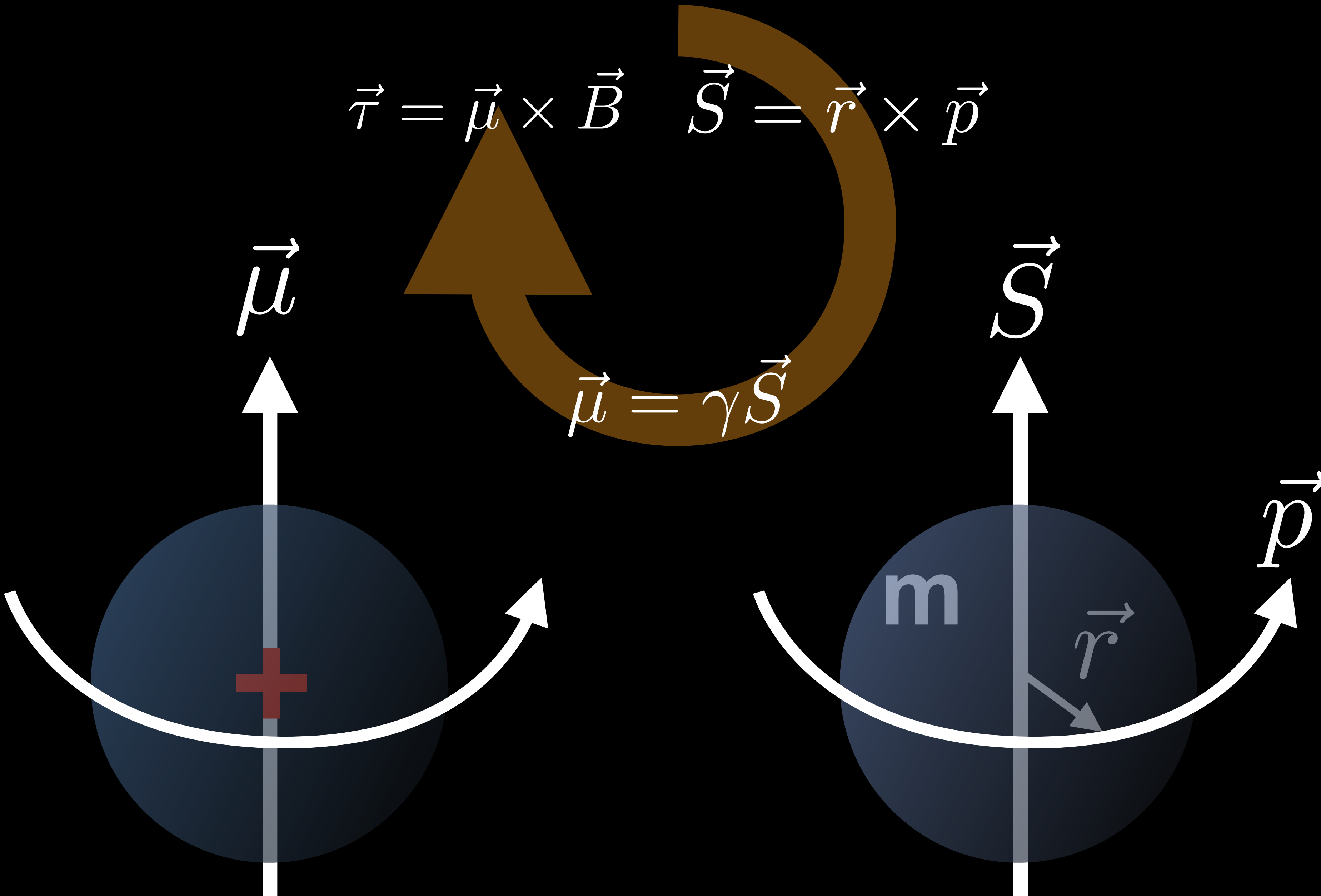


So where does the Larmor equation come from?

Magnetic Moments & Angular Momentum

$$\vec{\tau} = \vec{\mu} \times \vec{B} \quad \vec{S} = \vec{r} \times \vec{p}$$

$$\vec{\mu} = \gamma \vec{S}$$



Spin + Charge

Spin + Mass

Spin + Mass and Spin + Charge ➠ NMR

Equation of Motion for the Bulk Magnetization

$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}$$

Equation of motion for an ensemble of spins (isochromats) [Classical Description]

What is a general solution?

The *equation of motion* describes the bulk magnetization "behavior" in the presence of a B-field.

To the board...

Free & Forced Precession

Free vs. Forced Precession

<u>Free Precession</u> – Precession of the bulk magnetization vector about the static magnetic field after a pulse excitation. Free precession of the transverse magnetization at the Larmor frequency is responsible for the detectable NMR signal.

– *Liang & Lauterbur p. 375*

<u>Forced Precession</u> – Precession of the bulk magnetization about the excitation RF field.

– *Liang & Lauterbur p. 374*

Four Special Cases...

- **Laboratory Frame**
  - Coordinate system anchored to scanner
  - 1) *Free Precession* in the lab frame
  - 2) *Forced Precession* in the lab frame

- **Rotating Frame**
  - Coordinate system anchored to spin system
  - 3) *Free Precession* in the rotating frame
  - 4) *Forced Precession* in the rotating frame

- **...all without relaxation. We assume:**
  - a) Relaxation time constants are “really” long

  OR

  - b) Time scale of event is << relaxation time constant

Free Precession In The Laboratory Frame Without Relaxation

Rotations & Euler’s Formula

Vectors

- A ***vector*** ($\vec{v}$) describes a physical quantity (e.g. bulk magnetization or velocity) at a point in space and time and has a magnitude (positive real number), a direction, and physical units.

- To define a vector we need a ***basis***:

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

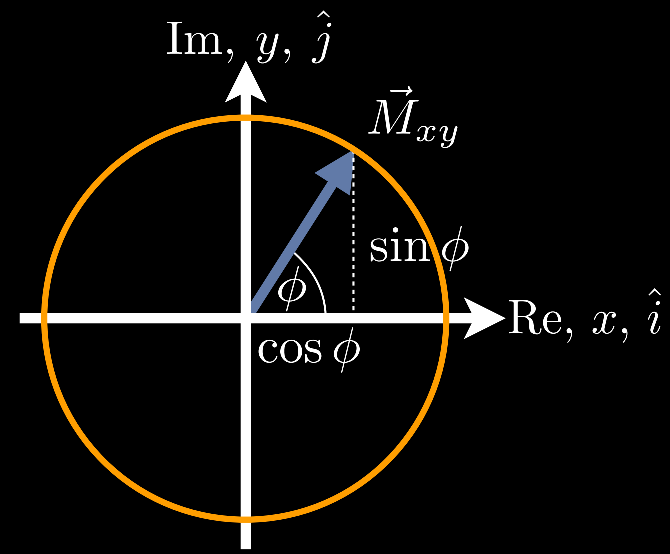
- A 3D ***vector*** has components:

$$\vec{M} = M_x\hat{i} + M_y\hat{j} + M_z\hat{k}$$

2D Vectors - Euler's Formula

- Euler's formula provides a compact representation of a 2D vector using a complex exponential:

$$e^{i\phi} = \cos\phi + i\sin\phi$$



$$
\begin{aligned}
\vec{M}_{xy} &= M_x\hat{i} + M_y\hat{j} && \text{Vector components}\\
&= M_x + iM_y && \text{Complex components}\\
&= |\vec{M}_{xy}|\cos\phi\hat{i} + |\vec{M}_{xy}|\sin\phi\hat{j} && \text{Trigonometric components}\\
&= |\vec{M}_{xy}|\cos\phi + i|\vec{M}_{xy}|\sin\phi && \text{Complex trigonometric components}\\
&= |\vec{M}_{xy}|e^{i\phi} && \text{Euler's notation}
\end{aligned}
$$

Euler's formula is mathematically convenient.
There is nothing explicitly *imaginary* about M_{xy}.

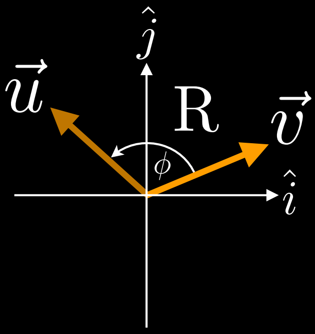
Rotations

- ***Rotations*** (R) are vector valued orthogonal transformations that preserve the magnitude of vectors and the angles between them.
- The simplest rotation matrix is the ***identity*** matrix:

$$\mathrm{R} = \mathrm{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{, therefore } \vec{v} = \mathrm{I}\vec{v}$$

- More simply, R transforms (rotates) one vector to another:

$$\vec{u} = \mathrm{R}\vec{v}$$



Rotations

$\hat{i}$ ends up here

$\hat{j}$ ends up here

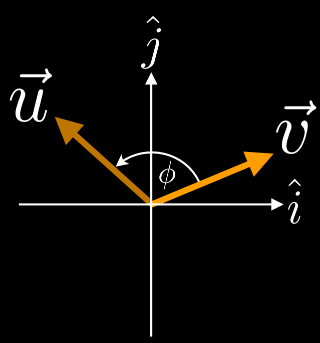
$\hat{k}$ does not change

Magnitude of rotation

$$R_z^{\phi} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Axis (phase) of rotation

$$\vec{u} = R\vec{v}$$



Note: Positive values of *Φ* produce right-handed (CCW) rotations.

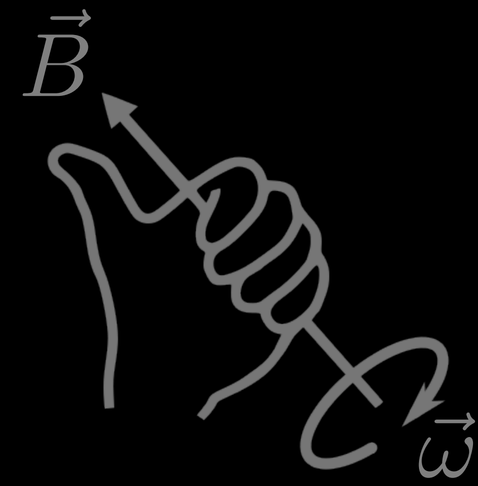
Free Precession In The Laboratory Frame Without Relaxation

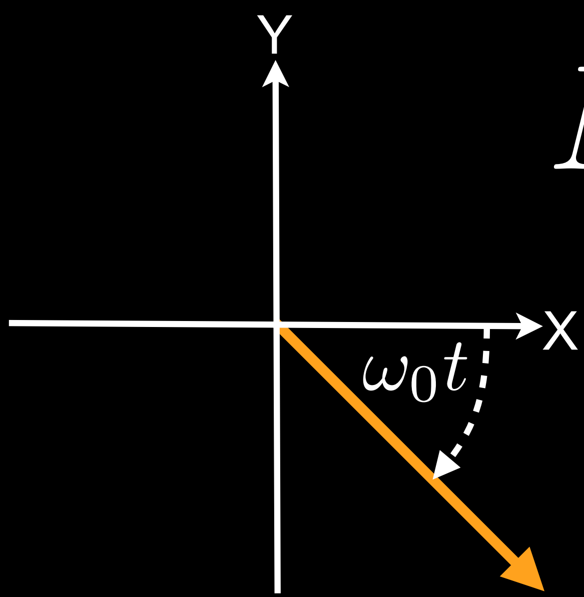
$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \left(\vec{B_0} \right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ M_x & M_y & M_z \\ 0 & 0 & \gamma B_0 \end{vmatrix}$$

To the board...

Free Precession In The Laboratory Frame Without Relaxation

$$\mathbf{R}_z(\omega_0 t) = \begin{bmatrix} \cos\omega_0 t & \sin\omega_0 t & 0 \\ -\sin\omega_0 t & \cos\omega_0 t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$





$$\vec{M}(t) = \mathbf{R}_z(\omega_0 t)\vec{M}^0$$

$$\Updownarrow$$

$$\omega_0 = \gamma B_0$$

Precession is left-handed (clockwise).

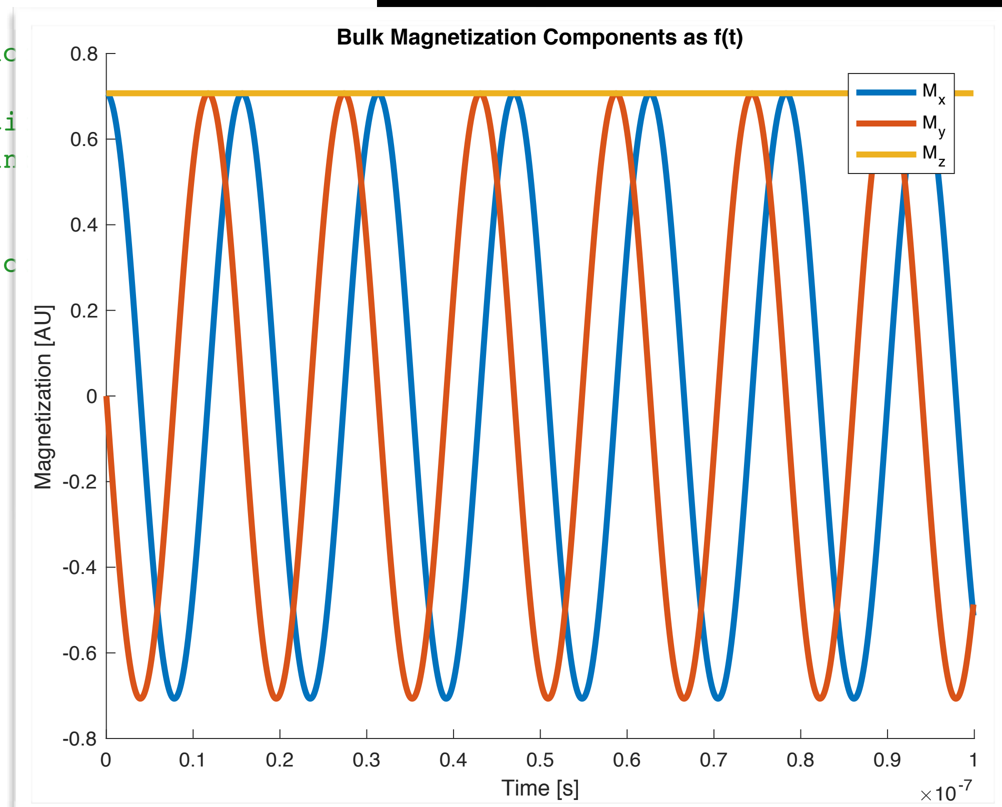
To The Board…

Matlab Example - Free Precession

```
% This function returns the 4x4 homogenous coordinate expression for
% precession for a particular gyromagnetic ratio (gamma), external
% field (B0), and time step (dt).
%
% SYNTAX:  dB0=PAM_B0_op(gamma,B0,dt)
%
% INPUTS:  gamma - Gyromagnetic ratio [Hz/T]
%          B0    - Main magnetic field [T]
%          dt    - Time step or vector [s]
%
% OUTPUTS: dB0   - Precessional operator [4x4]
%
% DBE@UCLA 01.21.2015

function dB0=PAM_B0_op(gamma,B0,dt)

if nargin==0
  gamma=42.57e6;         % Gyromagnetic ratio for 1H
  B0=1.5;                % Typical B0 field strength
  dt=ones(1,100)*1e-6;   % 100 1μs time steps
end

dB0=zeros(4,4,numel(dt));  % Initialize the array

for n=1:numel(dt)
  dw=2*pi*gamma*B0*dt(n);  % Incremental precession (rotation angle)

  % Precessional Operator (left handed)
  dB0(:,:,n)=[ cos(dw)  sin(dw)   0  0;
              -sin(dw)  cos(dw)   0  0;
                0         0       1  0;
                0         0       0  1];

end
return
```

Matlab Example - Free Precession

```
%% Filename: PAM_Lec02_B0_Free_Precession.m
%
% Demonstrate the precession of the bulk magnetization vector.
%
% DBE@UCLA 2017.12.20

%% Define some constants
gamma=42.57e6;                  % Gyromagnetic ratio for 1H [MHz/T]
B0=1.5;                         % B0 magnetic field strength [T]
dt=0.1e-9;                      % Time step [s]
nt=1000;                        % Number of time points to simulate
t=0:dt:((nt-1)*dt);             % Time vector [s]
M0=[sqrt(2)/2 0 sqrt(2)/2 1]';  % Arbitrary initial conditic

M=zeros(4,nt);                  % Initialize the magnetizati
M(:,1)=M0;                      % Define the first time poir

%% Simulate precession of the bulk magnetization vector
dB0=PAM_B0_op(gamma,B0,dt);     % Calculate the homogenous c

for n=2:nt
  M(:,n)=dB0*M(:,n-1);
end

%% Plot the results
figure; hold on;
  p(1)=plot(t,M(1,:));          % Plot the Mx component
  p(2)=plot(t,M(2,:));          % Plot the My component
  p(3)=plot(t,M(3,:));          % Plot the Mz component
    set(p,'LineWidth',3);       % Increase plot thickness
  ylabel('Magnetization [AU]');
  xlabel('Time [s]');
  legend('M_x','M_y','M_z');
  title('Bulk Magnetization Components as f(t)');
```



Note the number of cycles per ms.

Lecture 2 - Summary

$\vec{\tau} = \vec{\mu} \times \vec{B}$ $\quad \vec{S} = \vec{r} \times \vec{p}$

$\vec{\tau} = \frac{d\vec{S}}{dt}$ $\quad \vec{\mu} = \gamma \vec{S}$

$$\frac{d\vec{\mu}}{dt} = \vec{\mu} \times \gamma \vec{B}$$

Equation of Motion for a Magnetic Dipole

$$\vec{M} = \sum_{n=1}^{N_{total}} \vec{\mu}_n$$

$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}$$

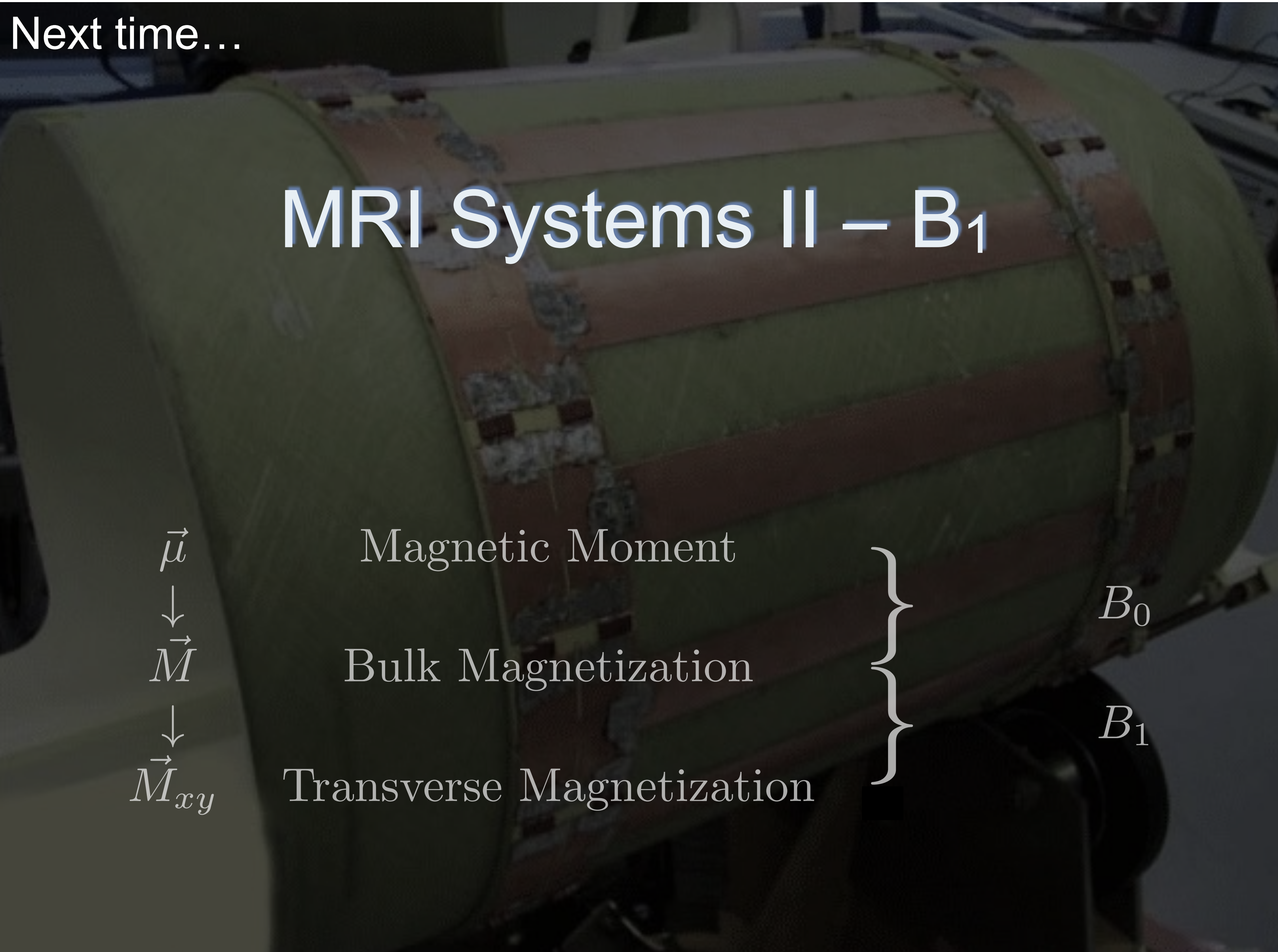
Equation of Motion for the bulk magnetization.

$$\vec{B}_0 = B_0 \vec{k}$$

$$\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \left(\vec{B}_0 \right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ M_x & M_y & M_z \\ 0 & 0 & \gamma B_0 \end{vmatrix}$$

$$M_x(t) = M_x^0 \cos(\gamma B_0 t) + M_y^0 \sin(\gamma B_0 t)$$
$$M_y(t) = -M_x^0 \sin(\gamma B_0 t) + M_y^0 \cos(\gamma B_0 t)$$
$$M_z(t) = M_z^0$$

Next time…



MRI Systems II – B_1

$$\vec{\mu} \quad \text{Magnetic Moment}$$
$$\downarrow$$
$$\vec{M} \quad \text{Bulk Magnetization}$$
$$\downarrow$$
$$\vec{M}_{xy} \quad \text{Transverse Magnetization}$$

$\vec{\mu} \rightarrow \vec{M}$: B_0

$\vec{M} \rightarrow \vec{M}_{xy}$: B_1

Thanks

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